

DEEP INELASTIC STRUCTURE FUNCTIONS AT TWO LOOPS

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Abstract

We present the analytic calculation of the Mellin moments of the structure functions F_2 , F_3 and F_L in perturbative QCD up to second order corrections and in leading twist approximation. We calculate the 2-loop contributions to the anomalous dimensions of the singlet and non-singlet operator matrix elements and the 2-loop coefficient functions of F_2 , F_3 and F_L . We perform the inverse Mellin transformation analytically and find our results in agreement with earlier calculations in the literature by Zijlstra and van Neerven.

1 Introduction

As one of the best studied reactions today, deep inelastic lepton-hadron scattering establishes the scale evolution of the structure functions, one of the most important precision tests of perturbative QCD. It provides unique information about the deep structure of the hadrons and most importantly, measurements of these structure functions in deep inelastic scattering (DIS) allow to extract the parton densities, which are subsequently used as input for many other hard scattering processes.

Over the years, the ever increasing accuracy of deep inelastic and other hard scattering experiments has created a steady demand for more accurate theoretical predictions. The qualitative quest for understanding scaling violations of the structure functions in terms of asymptotic freedom in QCD [1] soon changed to the quantitative task of reducing theoretical uncertainties due to higher order QCD corrections and the necessary calculations to obtain next-to-leading order (NLO) perturbative QCD predictions were performed in refs.[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

Today, high precision analyses of experimental data, such as the determination of the strong coupling constant α_s and the parton densities call for complete next-to-next-to-leading order (NNLO) perturbative QCD predictions. This has motivated the calculation of the 2-loop coefficient functions of all DIS structure functions by Zijlstra and van Neerven [13, 14, 15] and they could check their results for a number of fixed Mellin moments [16]. However, to complete the NNLO analysis of DIS structure functions one still has to know the perturbative QCD predictions at 3-loops for the anomalous dimensions of these structure functions and also the coefficient functions for the longitudinal structure function entering in the ratio of the longitudinal over the transverse cross section, $R = \sigma_L/\sigma_T$. These quantities are still unknown, except for a small number of fixed Mellin moments [17, 18, 19], some of them related to sum rules. The results of refs.[19], have already been used in NNLO analyses [19, 20]. Unfortunately, this limited information about some fixed Mellin moments is generally not sufficient to allow for NNLO analyses of all data of DIS and related hard scattering experiments, such as the Drell-Yan process. As a consequence, there are still considerable uncertainties on the parton densities, a prominent example being the gluon density at small x .

The aim of this paper is first of all to provide an independent check on the results of refs.[13, 14, 15], by means of a completely different method. At the same time, we also wish to demonstrate the power of our method, which we believe to be flexible enough and most promising in view of the ultimate challenge, the calculation of the anomalous dimensions at 3-loops. Our approach, that actually dates back to the origins of QCD [1, 4] is to calculate the Mellin moments of the DIS structure functions analytically as a general function of N . This idea was further pioneered by Kazakov and Kotikov [10] to obtain the longitudinal structure function F_L at two loops. In the present paper, we calculate in this way the Mellin moments of all unpolarized DIS structure functions F_2 , F_3 and F_L up to two loops. Subsequently, we perform the inverse Mellin transformation to express our results in momentum space as functions of x to compare with refs.[13, 14, 15].

A remark about other strategies towards completing the NNLO perturbative QCD predictions for DIS structure functions is in order here. A straightforward extension of the methods of ref.[19] to simply calculate sufficiently more fixed Mellin moments for a combined NNLO analyses of all data of, say, DIS and Drell-Yan experiments is not feasible. One has to know the Mellin moments at least up to $N = 100$ which would result in analytical expressions consisting of huge rational numbers. Unfortunately, this is beyond the capabilities of present computer algebra programs. A different approach is the direct calculation of the anomalous dimensions of DIS operator matrix elements, extending the work of ref.[5, 6] to 3-loops. First steps in this direction have been achieved in ref.[21], where the finite terms of DIS operator matrix elements at two loops have been calculated.

The outline of this paper is as follows. In section 2 we briefly discuss the operator product expansion (OPE) and some issues of the renormalization procedure. Section 3 gives a detailed explanation of the method to calculate the Mellin moments as an analytical function of N for a given diagram. It also contains a short summary of properties of harmonic sums [4, 22, 23] and harmonic polylogarithms [24]. Section 4 describes our calculation and lists our results for the structure functions F_2 , F_3 and F_L up

to two loops, both in Mellin space as a function of N and in momentum space as a function of x expressed in terms of harmonic polylogarithms. Finally, section 5 gives our conclusions. The appendices contain some relations between the standard polylogarithms and Nielsen functions and the harmonic polylogarithms and all explicit expressions for the 2-loop coefficient functions.

2 Formalism

In this section we set up the stage for the calculation of Mellin moments of the deep inelastic structure functions. We briefly recall the operator product expansion and, in particular, pay attention to the renormalization procedure. For reviews see refs.[25, 26].

We wish to calculate the hadronic part of the amplitude for unpolarized deep inelastic lepton-nucleon scattering which is given by the hadronic tensor

$$\begin{aligned} W_{\mu\nu}(p, q) &= \frac{1}{4\pi} \int d^4z e^{iq \cdot z} \langle P | J_\mu^\dagger(z) J_\nu(0) | P \rangle \\ &= e_{\mu\nu} \frac{1}{2x} F_L(x, Q^2) + d_{\mu\nu} \frac{1}{2x} F_2(x, Q^2) + i\epsilon_{\mu\nu\alpha\beta} \frac{p^\alpha q^\beta}{p \cdot q} F_3(x, Q^2), \end{aligned} \quad (1)$$

where J_μ is either an electromagnetic or a weak current and $|P\rangle$ is the unpolarized hadronic state. The boson transfers momentum q , $Q^2 = -q^2$, the hadron carries momentum p and the Bjorken scaling variable is defined as $x = Q^2/(2p \cdot q)$ with $0 < x \leq 1$. The tensors $e_{\mu\nu}$ and $d_{\mu\nu}$ are given by

$$e_{\mu\nu} = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}, \quad (2)$$

$$d_{\mu\nu} = -g_{\mu\nu} - p_\mu p_\nu \frac{4x^2}{q^2} - (p_\mu q_\nu + p_\nu q_\mu) \frac{2x}{q^2}. \quad (3)$$

The longitudinal structure function F_L is related to the structure function F_1 ,

$$F_L(x, Q^2) = F_2(x, Q^2) - 2xF_1(x, Q^2). \quad (4)$$

The structure function F_3 describes parity-violating effects that arise from vector and axial-vector interference. It vanishes for pure electromagnetic interactions.

We are interested in the Mellin moments of the structure functions, defined as

$$F_i^N(Q^2) = \int_0^1 dx x^{N-2} F_i(x, Q^2), \quad i = 2, L. \quad (5)$$

A similar relation defines F_3^N with N replaced by $N+1$ in the integral on the right hand side in eq.(5).

2.1 Operator product expansion

In the Bjorken limit, $Q^2 \rightarrow \infty$, x fixed, the integral in eq.(1) is dominated by the integration region near the lightcone $z^2 \sim 0$, because in this region the phase in the exponent in eq.(1) becomes stationary. The external momentum q of the hadronic scattering amplitude is highly virtual, that is to say in the unphysical Euclidean region. Thus, we can use dispersion relations together with the operator product expansion for a formal expansion of the current product in eq.(1) around the lightcone $z^2 \sim 0$ into a series of local composite operators of leading twist.

The optical theorem relates the hadronic tensor in eq.(1) to the imaginary part of the forward scattering amplitude of boson-nucleon scattering, $T_{\mu\nu}$,

$$W_{\mu\nu}(p, q) = \frac{1}{2\pi} \text{Im } T_{\mu\nu}(p, q). \quad (6)$$

The forward Compton amplitude $T_{\mu\nu}$ has a time-ordered product of two local currents, to which standard perturbation theory applies,

$$T_{\mu\nu}(p, q) = i \int d^4 z e^{iq \cdot z} \langle P | T \left(J_\mu^\dagger(z) J_\nu(0) \right) | P \rangle. \quad (7)$$

In terms of local operators for a time ordered product of the two electromagnetic or weak hadronic currents such as in eq.(7) the OPE reads

$$\begin{aligned} i \int d^4 z e^{iq \cdot z} T \left(J_{\nu_1}^\dagger(z) J_{\nu_2}(0) \right) = & \quad (8) \\ & \sum_{N,j} \left(\frac{1}{Q^2} \right)^N \left[\left(g_{\nu_1 \nu_2} - \frac{q_{\nu_1} q_{\nu_2}}{q^2} \right) q_{\mu_1} q_{\mu_2} C_{L,j}^N \left(\frac{Q^2}{\mu^2}, \alpha_s \right) \right. \\ & - \left(g_{\nu_1 \mu_1} g_{\nu_2 \mu_2} q^2 - g_{\nu_1 \mu_1} q_{\nu_2} q_{\mu_2} - g_{\nu_2 \mu_2} q_{\nu_1} q_{\mu_1} + g_{\nu_1 \nu_2} q_{\mu_1} q_{\mu_2} \right) C_{2,j}^N \left(\frac{Q^2}{\mu^2}, \alpha_s \right) \\ & \left. + i \epsilon_{\nu_1 \nu_2 \mu_1 \nu_3} g^{\nu_3 \nu_4} q_{\nu_4} q_{\mu_2} C_{3,j}^N \left(\frac{Q^2}{\mu^2}, \alpha_s \right) \right] q_{\mu_3} \dots q_{\mu_N} O^{j, \{\mu_1, \dots, \mu_N\}}(\mu^2) + \text{higher twists}, \end{aligned}$$

where $j = \alpha, q, g$. Here, all quantities are assumed to be renormalized, μ being the renormalization scale. Higher twist contributions are less singular near the lightcone $z^2 \sim 0$ and suppressed by powers of $1/Q^2$. They are omitted in eq.(8). Thus, the sum over N in eq.(8) extends to infinity and runs only over the standard set of the spin- N twist-2 irreducible, symmetrical and traceless flavour non-singlet quark operators O^α , and the singlet quark and gluon operators O^q and O^g . These are defined by,

$$O^{\alpha, \{\mu_1, \dots, \mu_N\}} = \bar{\psi} \lambda^\alpha \gamma^{\{\mu_1} D^{\mu_2} \dots D^{\mu_N\}} \psi, \quad \alpha = 1, 2, \dots, (n_f^2 - 1), \quad (9)$$

$$O^q, \{\mu_1, \dots, \mu_N\} = \bar{\psi} \gamma^{\{\mu_1} D^{\mu_2} \dots D^{\mu_N\}} \psi, \quad (10)$$

$$O^g, \{\mu_1, \dots, \mu_N\} = F^{\{\nu \mu_1} D^{\mu_2} \dots D^{\mu_{N-1}} F^{\mu_N \nu\}}. \quad (11)$$

Here, ψ defines the quark operator and $F^{\mu\nu}$ the gluon operator. The generators of the flavour group $SU(n_f)$ are denoted by λ^α , and the covariant derivative by D^μ . It is understood that the symmetrical and traceless part is taken with respect to the indices in curly brackets.

The spin averaged matrix elements of these operators in eqs.(9)–(11) sandwiched between some hadronic state are given by

$$\langle P | O^{j, \{\mu_1, \dots, \mu_N\}} | P \rangle = p^{\{\mu_1} \dots p^{\mu_N\}} A_{P,N}^j \left(\frac{p^2}{\mu^2} \right), \quad (12)$$

where hadron mass effects have been neglected. The anomalous dimensions of these operator matrix elements in eq.(12) govern the scale evolution of the structure functions. They are finite quantities as well as the coefficient functions $C_{i,j}^N$ multiplying these operator matrix elements according to eq.(8). Both are calculable order by order in perturbative QCD in an expansion in the strong coupling constant α_s .

The operator matrix elements themselves as given in eq.(12) are not calculable in perturbative QCD. However, they can be related to the distributions q_i , \bar{q}_i of quark and anti-quark of flavour i and to the gluon distribution g in the hadron. Let us briefly note, that the general structure of these densities allows for three independently evolving types of non-singlet distributions $q_{ns,ij}^\pm$ and q_{ns}^V and one quark singlet distribution q_s . The three non-singlet distributions are the flavour asymmetries

$$q_{ns,ij}^\pm = q_i \pm \bar{q}_i - (q_j \pm \bar{q}_j), \quad (13)$$

and the sum of the valence distributions of all flavours,

$$q_{ns}^V = \sum_{i=1}^{n_f} (q_i - \bar{q}_i), \quad (14)$$

while the singlet distribution is simply the sum of the distributions of all flavours,

$$q_s = \sum_{i=1}^{n_f} (q_i + \bar{q}_i). \quad (15)$$

With the operator matrix elements in eq.(12) we can write the OPE of eq.(8) as

$$\begin{aligned} i \int d^4 z e^{iq \cdot z} \langle P | T \left(J_\mu^\dagger(z) J_\nu(0) \right) | P \rangle = & \quad (16) \\ \sum_{N,j} \left(\frac{1}{2x} \right)^N \left[e_{\mu\nu} C_{L,j}^N \left(\frac{Q^2}{\mu^2}, \alpha_s \right) + d_{\mu\nu} C_{2,j}^N \left(\frac{Q^2}{\mu^2}, \alpha_s \right) + i \epsilon_{\mu\nu\alpha\beta} \frac{p^\alpha q^\beta}{p \cdot q} C_{3,j}^N \left(\frac{Q^2}{\mu^2}, \alpha_s \right) \right] A_{P,N}^j \left(\frac{p^2}{\mu^2} \right) \\ + \text{higher twists,} \end{aligned}$$

which is an expansion in terms of the variable $(2p \cdot q)/Q^2 = 1/x$ for unphysical $x \rightarrow \infty$. The final connection to DIS structure functions in the physical region $0 < x \leq 1$ is achieved by taking Mellin moments of eq.(16) and using the optical theorem eq.(6), which relates the structure functions to invariants $T_{i,P}$, $i = 2, 3, L$, of the forward Compton amplitude $T_{\mu\nu}$ as follows,

$$\frac{1}{2\pi} \text{Im} T_{i,P}(x, Q^2) = \frac{1}{2x} F_i(x, Q^2), \quad i = 2, L, \quad (17)$$

$$\frac{1}{2\pi} \text{Im} T_{3,P}(x, Q^2) = F_3(x, Q^2), \quad (18)$$

where the $T_{i,P}$ can be projected in leading twist approximation and $D = 4 - 2\epsilon$ dimensions,

$$T_{L,P}(x, Q^2) = -\frac{q^2}{(p \cdot q)^2} p^\mu p^\nu T_{\mu\nu}(p, q), \quad (19)$$

$$T_{2,P}(x, Q^2) = -\left(\frac{3-2\epsilon}{2-2\epsilon} \frac{q^2}{(p \cdot q)^2} p^\mu p^\nu + \frac{1}{2-2\epsilon} g^{\mu\nu} \right) T_{\mu\nu}(p, q), \quad (20)$$

$$T_{3,P}(x, Q^2) = -i \frac{1}{(1-2\epsilon)(2-2\epsilon)} \epsilon^{\mu\nu\alpha\beta} \frac{p_\alpha q_\beta}{p \cdot q} T_{\mu\nu}(p, q). \quad (21)$$

Then, with the help of eqs.(16)–(18) we find

$$\int_0^1 dx x^{N-2} F_i(x, Q^2) = \sum_{j=\alpha, q, g} C_{i,j}^N \left(\frac{Q^2}{\mu^2}, \alpha_s \right) A_{P,N}^j \left(\frac{p^2}{\mu^2} \right), \quad i = 2, L, \quad (22)$$

$$\int_0^1 dx x^{N-1} F_3(x, Q^2) = \sum_{j=\alpha} C_{3,j}^N \left(\frac{Q^2}{\mu^2}, \alpha_s \right) A_{P,N}^j \left(\frac{p^2}{\mu^2} \right), \quad (23)$$

which shows that the Mellin moments of DIS structure functions F_i^N as defined in eq.(5) can naturally be written in the parameters of the OPE eq.(8).

The derivation of eqs.(22) and (23) in the dispersive approach uses symmetry properties of $T_{\mu\nu}$ under exchange $q \rightarrow -q$, which is $x \rightarrow -x$. Therefore, dependent on the process under consideration, eqs.(22) and (23) determine only either the even or the odd Mellin moments of F_2 , F_L and F_3 . However, all moments in the complex N plane are fixed by analytic continuation from either the even or the odd Mellin moments. This implies that the x -space result for the physical structure functions in the range $0 < x \leq 1$, can be found by means of an inverse Mellin transformation if the infinite set of either even or odd moments is known.

In the standard case of unpolarized electron-proton scattering in the one-photon exchange approximation the even moments of F_2 and F_L are fixed in eq.(22), while in the case of electroweak

interactions with neutrino-proton scattering eqs.(22) and (23) determine the odd moments of $F_2^{\nu P-\bar{\nu}P}$ and $F_3^{\nu P+\bar{\nu}P}$ and the even moments of $F_2^{\nu P+\bar{\nu}P}$ and $F_3^{\nu P-\bar{\nu}P}$, see ref.[25]. We are going to consider all these cases mentioned in order to extract the complete non-singlet coefficient functions at two loops, as will be detailed in section 4.

The sum in eq.(22) extends over the flavour non-singlet and singlet quark and gluon contributions. Notice however, that in eq.(23) the singlet operators O^q , O^g do not contribute to F_3 . This is due to the properties of O^q and O^g under a charge conjugation transformation [25]. It can also be understood by looking at partonic scattering processes. If p and \bar{p} are partons and anti-partons, charge conjugation implies for the cross-sections

$$\sigma_p = -\sigma_{\bar{p}}, \quad (24)$$

which gives zero for eigenstates under charge conjugation [15].

2.2 Renormalization

We are interested in the calculation of the scale evolution of the DIS structure functions. To that end, a short discussion of the renormalization properties of the operators and the coefficient functions in eq.(8) is in order.

Let us first recall, that the OPE of eq.(8) is an operator statement and therefore both the coefficient functions $C_{i,j}^N$ and the anomalous dimensions of the operators do not depend on the hadronic states to which the OPE is applied. The information on the hadronic target is only contained in the operator matrix elements $A_{p,N}^j$ in eq.(12). It is therefore standard to consider simpler Green's functions in an infrared regulated perturbative expansion with the operators O^j sandwiched between parton states

$$\langle p | O^{j, \{\mu_1, \dots, \mu_N\}} | p \rangle = p^{\{\mu_1 \dots \mu_N\}} A_{p,N}^j \left(\frac{p^2}{\mu^2} \right), \quad (25)$$

where $|p\rangle$ denotes a spin-averaged parton state, being either a flavour non-singlet or singlet combination of quarks and anti-quarks or a gluon.

As they stand the bare operator matrix elements in eq.(25) require renormalization. We choose dimensional regularization [27, 28] in $D = 4 - 2\epsilon$ dimensions and define the renormalized operators in terms of bare operators as

$$O^{\alpha, \text{bare}} = Z^{\alpha\alpha} O^{\alpha, \text{ren}}, \quad (26)$$

$$O^{j, \text{bare}} = \sum_{k=q,g} Z^{jk} O^{k, \text{ren}}, \quad j = q, g, \quad (27)$$

where $Z^{\alpha\alpha}$ renormalizes the quark flavour non-singlet operator O^α . In the flavour singlet case, eq.(27) denotes operator mixing under renormalization as O^q and O^g have the same quantum numbers.

The anomalous dimensions γ determine the scale dependence of the renormalized operators,

$$\frac{d}{d \ln \mu^2} O^{\text{ren}} \equiv \gamma O^{\text{ren}}, \quad (28)$$

and they are defined as

$$\gamma = \left(\frac{d}{d \ln \mu^2} Z \right) Z^{-1}, \quad (29)$$

where in the flavour singlet case eqs.(28) and (29) are understood as matrix equations, Z representing the matrix Z^{jk} and γ the matrix γ_{jk} . The general structure of these anomalous dimensions is constrained by charge conjugation invariance and flavour symmetry. In the case of quarks and anti-quarks of flavour i, j they can be split up into a valence and a sea part,

$$\gamma_{q_i q_j} = \gamma_{\bar{q}_i \bar{q}_j} = \delta_{ij} \gamma_{qq}^V + \gamma_{qq}^S, \quad (30)$$

$$\gamma_{q_i \bar{q}_j} = \gamma_{\bar{q}_i q_j} = \delta_{ij} \gamma_{q\bar{q}}^V + \gamma_{q\bar{q}}^S. \quad (31)$$

As one commonly considers matrix elements of operators O^j corresponding to the quark non-singlet and singlet distributions $q_{ns,ij}^\pm$, q_{ns}^V and q_s of eqs.(13)–(15), we remark that their scale evolution is governed by the following four linear combinations,

$$q_{ns,ij}^\pm \longrightarrow \delta_{ij} \gamma_{qq}^{\pm,V} = \delta_{ij} (\gamma_{qq}^V \pm \gamma_{q\bar{q}}^V) \quad (32)$$

$$q_{ns}^V \longrightarrow \gamma_{q\bar{q}}^{-,V} + n_f \gamma_{q\bar{q}}^{-,S} = \gamma_{qq}^V - \gamma_{q\bar{q}}^V + n_f (\gamma_{qq}^S - \gamma_{q\bar{q}}^S), \quad (33)$$

$$q_s \longrightarrow \gamma_{qq}^{+,V} + n_f \gamma_{qq}^{+,S} = \gamma_{qq}^V + \gamma_{q\bar{q}}^V + n_f (\gamma_{qq}^S + \gamma_{q\bar{q}}^S). \quad (34)$$

In our case of unpolarized lepton-hadron scattering, the even moments of F_2 determine the linear combinations $\gamma_{qq}^{+,V}$ and $\gamma_{qq}^{+,S}$, while the odd moments of F_3 determine $\gamma_{q\bar{q}}^{-,V}$ and $\gamma_{q\bar{q}}^{-,S}$. The individual valence and sea contributions are identified by the flavour structure of the diagrams for the structure functions, so that eqs.(32)–(34) suffice to determine the anomalous dimensions $\gamma_{qq}^V, \gamma_{q\bar{q}}^V, \gamma_{qq}^S$ and $\gamma_{q\bar{q}}^S$.

Both, the anomalous dimensions γ and the renormalization constants Z have a series expansion in α_s . We rewrite eq.(29) in $D = 4 - 2\epsilon$ dimensions in a form that is suitable for an easy extraction of the expansion coefficients,

$$\gamma(\alpha_s) = \left(\epsilon \frac{\alpha_s}{4\pi} - \beta(\alpha_s) \right) \left(\frac{d}{d \ln \mu^2} Z \left(\alpha_s, \frac{1}{\epsilon} \right) \right) Z^{-1} \left(\alpha_s, \frac{1}{\epsilon} \right), \quad (35)$$

where $\beta(\alpha_s)$ denotes the beta function, that determines the renormalization scale dependence of the running coupling. The renormalization constants Z obey an expansion in $1/\epsilon$. Although the Z contain poles in ϵ , the anomalous dimensions are always finite as $\epsilon \rightarrow 0$. Thus, to lowest order in α_s the anomalous dimensions γ in eq.(35) are simply expressed through the residue of Z in $1/\epsilon$.

To solve the coupled matrix equation defined by eq.(29) in the singlet case, one should notice that to leading order in α_s the matrix Z^{jk} is diagonal with $Z^{(0),qq} = Z^{(0),gg} = 1$ and $Z^{(0),qg} = Z^{(0),gq} = 0$. This additional information allows for a unique iterative determination of the anomalous dimensions order by order in α_s .

The relation between the bare coupling α_s^{bare} and the renormalized coupling α_s is given by

$$\alpha_s^{\text{bare}} = Z_{\alpha_s} \alpha_s^{\text{ren}}, \quad (36)$$

with the renormalization constant Z_{α_s} in the minimal subtraction scheme given by

$$Z_{\alpha_s} = 1 - \frac{\beta_0}{\epsilon} \frac{\alpha_s}{4\pi} + \left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon} \right) \left(\frac{\alpha_s}{4\pi} \right)^2 + \dots \quad (37)$$

In $D = 4 - 2\epsilon$ dimensions the beta function of the running coupling up to two loops [29] is given by

$$\begin{aligned} \frac{d\alpha_s/(4\pi)}{d \ln \mu^2} &= -\epsilon \frac{\alpha_s}{4\pi} - \beta_0 \left(\frac{\alpha_s}{4\pi} \right)^2 - \beta_1 \left(\frac{\alpha_s}{4\pi} \right)^3 - \dots, \\ \beta_0 &= \frac{11}{3} C_A - \frac{4}{3} T_F n_f, \\ \beta_1 &= \frac{34}{3} C_A^2 - 4 C_F T_F n_f - \frac{20}{3} C_A T_F n_f. \end{aligned} \quad (38)$$

Finally, to determine the scale dependence of the coefficient functions it is sufficient to notice, that the anomalous dimension of the current J_μ in the time-ordered product of the forward Compton amplitude $T_{\mu\nu}$ is zero due to current conservation. It implies, that the scale evolution of the coefficient functions and the operator matrix elements is governed by the same anomalous dimensions, which provides us with the renormalization group equation for the flavour singlet and non-singlet coefficient

functions,

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s(\mu^2)) \frac{\partial}{\partial \alpha_s(\mu^2)} - \gamma_{\text{qq}}^{\text{ns}}(\alpha_s(\mu^2)) \right] C_{i,q}^{N,\text{ns}} \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) = 0, \quad (39)$$

$$\sum_{k=\text{q,g}} \left[\left\{ \mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s(\mu^2)) \frac{\partial}{\partial \alpha_s(\mu^2)} \right\} \delta_{jk} - \gamma_{jk}(\alpha_s(\mu^2)) \right] C_{i,k}^N \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) = 0, \quad (40)$$

$$j = \text{q,g}.$$

In eq.(39) we have adopted the conventional notation to collectively denote the non-singlet anomalous dimensions of eqs.(32) and (33) with $\gamma_{\text{qq}}^{\text{ns}}$ and the coefficient functions with $C_{i,q}^{N,\text{ns}}$. In particular, the Q^2 -dependence of the coefficient functions does not depend on the index α of the non-singlet operator O^α in eq.(9) anymore, see for instance ref.[30].

The actual calculation of the anomalous dimensions as defined in eq.(29) and the coefficient functions $C_{i,j}^N$ in perturbative QCD proceeds as follows. We introduce partonic invariants $T_{i,p}$, $i = 2, 3, L$, of the forward partonic Compton amplitude in analogy to eqs.(19)–(21). By means of the OPE eq.(8), these invariants can be written in terms of renormalized operator matrix elements as

$$T_{i,q}^{\text{ns}}(x, Q^2, \alpha_s, \epsilon) = \quad (41)$$

$$\sum_N \left(\frac{1}{2x} \right)^N C_{i,j}^{N,\text{ns}} \left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon \right) Z_{\text{ns}}^{\text{qq}} \left(\alpha_s, \frac{1}{\epsilon} \right) A_{\text{q},N}^{\text{ns,ren}} \left(\alpha_s, \frac{p^2}{\mu^2}, \epsilon \right) + O(p^2),$$

$$T_{i,p}(x, Q^2, \alpha_s, \epsilon) = \quad (42)$$

$$\sum_N \sum_{j,k=\text{q,g}} \left(\frac{1}{2x} \right)^N C_{i,j}^N \left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon \right) Z^{jk} \left(\alpha_s, \frac{1}{\epsilon} \right) A_{\text{p},N}^{k,\text{ren}} \left(\alpha_s, \frac{p^2}{\mu^2}, \epsilon \right) + O(p^2), \quad \text{p} = \text{q,g},$$

where we have distinguished the flavour non-singlet from the flavour singlet case, $i = 2, 3, L$. In the former case, we used again the collective notation $Z_{\text{ns}}^{\text{qq}}$ for the various non-singlet renormalization constants. The left hand side of eqs.(41) and (42) is renormalized by substituting the bare coupling constant in terms of the renormalized one as defined by eq.(36), $\alpha_s = \alpha_s(\mu^2/\Lambda^2)$. The wave function renormalization factors for the external quark and gluon lines are overall factors on both sides of the equations and drop out. The terms $O(p^2)$ on the right hand side of eqs.(41) and (42) indicate higher twist contributions, which we neglect.

It is known that the gauge invariant operators O^q and O^g mix under renormalization with unphysical operators [7, 31]. These are BRST variations of some operators or else vanish by the equations of motion. However, matrix elements with physical polarization and on-shell momenta of such unphysical operators vanish. That is to say, these unphysical operators do not contribute to quantities related to physical S -matrix elements such as the invariants $T_{i,p}$ which we are going to calculate. Therefore, they are omitted in eq.(42).

Starting with the partonic invariants $T_{i,p}$ from eqs.(41) and (42), the renormalization constants Z and the coefficient functions $C_{i,j}^N$ are calculated using the method of projection developed in ref.[32]. This method consists of applying the following projection operator to both sides of eqs.(41) and (42)

$$\mathcal{P}_N \equiv \left[\frac{q^{\{\mu_1 \dots \mu_N\}}}{N!} \frac{\partial^N}{\partial p^{\mu_1} \dots \partial p^{\mu_N}} \right] \Big|_{p=0}, \quad (43)$$

where $q^{\{\mu_1 \dots \mu_N\}}$ is the harmonic, that is to say the symmetrical and traceless part of the tensor $q^{\mu_1} \dots q^{\mu_N}$.

On the right hand side of eqs.(41) and (42), it is obvious, that the N -th order differentiation in the projection operator \mathcal{P}_N singles out precisely the N -th moment which is the coefficient of $1/(2x)^N$.

All other powers of $1/(2x)$ vanish either by differentiation or after nullifying the momentum p . The operator \mathcal{P}_N does not act on the renormalization constants Z and the coefficient functions on the right hand side of eqs.(41) and (42) as they are only functions of N , α_s and ϵ . However, \mathcal{P}_N does act on the partonic matrix elements $A_{p,N}^j$. There, the nullification of p effectively eliminates all diagrams containing loops, which become massless tadpole diagrams and are therefore put to zero in dimensional regularization. In particular, this removes the operator matrix elements $A_{g,N}^{q,\text{ren}}$ and $A_{q,N}^{g,\text{ren}}$ in eq.(42) as they start only at 1-loop level. Hence, in the perturbative expansion of $A_{p,N}^j$ only the tree level diagrams $A_{p,N}^{p,\text{tree}}$ survive. Finally, the $O(p^2)$ terms in eqs.(41) and (42), which denote higher twist contributions, become proportional to the metric tensor after differentiation. They are removed by the harmonic tensor $q^{\{\mu_1 \dots \mu_N\}}$.

On the left hand side of eqs.(41) and (42), \mathcal{P}_N is applied to the integrands of all Feynman diagrams contributing to the invariants $T_{i,p}$. The momentum p is nullified before taking the limit $\epsilon \rightarrow 0$, so that all infrared divergences as $p \rightarrow 0$ are dimensionally regularized for individual diagrams. Effectively this reduces the 4-point diagrams that contribute to $T_{\mu\nu}$ to 2-point diagrams with symbolic powers of scalar products in the numerator and denominator, which we can solve by means of recursion relations as will be detailed in section 3. Notice also, that we apply \mathcal{P}_N to the projected partonic invariants $T_{i,p}$ rather than to $T_{\mu\nu}$, as \mathcal{P}_N would destroy the tensor structure of $T_{\mu\nu}$.

To summarize, we find after application of the projection operator \mathcal{P}_N to eqs.(41) and (42),

$$T_{i,q}^{N,\text{ns}} \left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon \right) = C_{i,q}^{N,\text{ns}} \left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon \right) Z_{\text{ns}}^{\text{qq}} \left(\alpha_s, \frac{1}{\epsilon} \right) A_{q,N}^{\text{ns,tree}}(\epsilon), \quad (44)$$

$$T_{i,p}^N \left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon \right) = \left[C_{i,q}^N \left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon \right) Z^{\text{qp}} \left(\alpha_s, \frac{1}{\epsilon} \right) + C_{i,g}^N \left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon \right) Z^{\text{gp}} \left(\alpha_s, \frac{1}{\epsilon} \right) \right] A_{p,N}^{p,\text{tree}}(\epsilon), \quad p = q, g, \quad (45)$$

where $i = 2, 3, L$ and the left hand side is defined as

$$T_{i,p}^N \left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon \right) \equiv \mathcal{P}_N T_{i,p}(x, Q^2, \alpha_s, \epsilon) \Big|_{p=0}. \quad (46)$$

Eqs.(44) and (45) are central to our approach of the calculation of the anomalous dimensions and coefficient functions via the OPE and dispersion relations. They represent a coupled system of equations when both sides are expanded in powers of α_s and ϵ . That is to say, the $C_{i,j}^N$ are expanded in positive powers of ϵ and the Z are expanded in negative powers of ϵ . Explicit solutions to eqs.(44) and (45) in terms of anomalous dimensions and coefficient functions will be given in section 4.

At this point a few remarks are in order. First of all, eq.(45) does not provide us with the full information about the renormalization constants, because Z^{gq} and Z^{gg} are determined only in the order α_s . This limitation follows directly from the fact that $C_{i,g}^N$ only starts from order α_s since the photon couples directly only to quarks. To extract the 2-loop anomalous dimension $\gamma_{\text{gq}}^{(1)}$ and $\gamma_{\text{gg}}^{(1)}$ we also calculate Green's functions in which the photon is replaced by an external scalar particle ϕ that couples directly only to gluons [19].

These Green's functions can be expressed in partonic invariants $T_{\phi,p}$, for which an OPE similar to eq.(8) exists with the same singlet operators O^q and O^g but with different coefficient functions $C_{\phi,p}^N$. The important point here is that now $C_{\phi,g}^N$ starts already at order α_s^0 . Hence, the $T_{\phi,p}$, provide us with the necessary renormalization constants Z^{gq} and Z^{gg} of the singlet operators to two loops. The vertices that describe the coupling between the external scalar field ϕ and the gluons can be obtained by adding the simplest gauge invariant interaction term $\phi F_{\mu\nu}^a F_a^{\mu\nu}$ to the QCD Lagrangian, where $F_{\mu\nu}^a$ is the QCD field strength.

Repeating the steps that led to eq.(45) one finds for these partonic invariants $T_{\phi,p}$,

$$(Z_{F^2})^{-2} T_{\phi,p}^N \left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon \right) = \left[C_{\phi,q}^N \left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon \right) Z^{\text{qp}} \left(\alpha_s, \frac{1}{\epsilon} \right) + C_{\phi,g}^N \left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon \right) Z^{\text{gp}} \left(\alpha_s, \frac{1}{\epsilon} \right) \right] A_{p,N}^{\text{p,tree}}(\epsilon), \quad p = q, g. \quad (47)$$

Beyond tree level we have to take into account the overall renormalization [33] of the operator $\phi F^{a\mu\nu} F_{\mu\nu}^a$ with the renormalization constant Z_{F^2} ,

$$(F^{a\mu\nu} F_{\mu\nu}^a)^{\text{bare}} = Z_{F^2} (F^{a\mu\nu} F_{\mu\nu}^a)^{\text{ren}} + \dots, \quad Z_{F^2} = \frac{1}{1 - \beta(\alpha_s)/(\epsilon\alpha_s)}. \quad (48)$$

For the partonic invariants $T_{\phi,p}^N$ of the scalar particle ϕ this implies an additional overall renormalization

$$(T_{\phi,p}^N)^{\text{ren}} = (Z_{F^2})^{-2} (T_{\phi,p}^N)^{\text{bare}}, \quad (49)$$

as indicated on the left hand side in eq.(47). The dots in eq.(48) indicate mixing with unphysical operators, which give vanishing contributions to on-shell matrix elements with physical spin projections. The only physical operator, that mixes with $\phi F^{a\mu\nu} F_{\mu\nu}^a$ under renormalization is a quark mass term, $m_q \bar{\psi}\psi$, which vanishes in the limit of massless quarks.

A second remark concerns the parity-violating structure function F_3 , which is obtained from the time-ordered product of one vector current V_μ and one axial vector current A_ν . The axial current contains a γ_5 coupling and it is well known that this requires some care in the framework of dimensional regularization. We define the axial current by the substitution [34]

$$\gamma_\mu \gamma_5 = i \frac{1}{6} \epsilon_{\mu\rho\sigma\tau} \gamma^\rho \gamma^\sigma \gamma^\tau, \quad (50)$$

from which F_3 is projected according to eqs.(18) and (21). The sum over dummy Lorentz indices in the projection such as in $\epsilon^{\mu\rho\sigma\tau} \epsilon_{\mu\alpha\beta\chi}$ or $\epsilon^{\mu\nu\rho\sigma} \epsilon_{\mu\nu\alpha\beta}$ defines products of metric tensors, which have to be considered as D -dimensional objects [18, 35].

The definition eq.(50) violates the axial Ward identity which is to be restored by an additional renormalization. The necessary renormalization constant Z_A in the minimal subtraction scheme has been given in ref.[18]. Up to two loops it is,

$$Z_A = 1 + \left(\frac{\alpha_s}{4\pi} \right)^2 \frac{1}{\epsilon} \left\{ \frac{22}{3} C_A C_F - \frac{4}{3} C_F n_f \right\}, \quad (51)$$

where the expansion is performed in terms of the renormalized coupling α_s at the scale μ^2 .

In addition, the treatment of γ_5 in $D = 4 - 2\epsilon$ dimensions introduces an extra finite renormalization with Z_5 . This is derived in the minimal subtraction scheme from an obvious relation between the vector and the axial-vector current,

$$(R_{\overline{\text{MS}}} V_\mu) \gamma_5 = Z_5 R_{\overline{\text{MS}}} A_\mu, \quad (52)$$

where $R_{\overline{\text{MS}}}$ denotes the R -operation in the $\overline{\text{MS}}$ -scheme to remove ultraviolet divergencies. It has also been given in ref.[18] and reads up to two loops,

$$Z_5 = 1 - \frac{\alpha_s}{4\pi} 4 C_F + \left(\frac{\alpha_s}{4\pi} \right)^2 \left[22 C_F^2 - \frac{107}{9} C_A C_F + \frac{2}{9} C_F n_f \right], \quad (53)$$

where again the expansion is performed in the renormalized coupling α_s at the scale μ^2 . These additional renormalizations in eqs.(51) and (53) have to be taken into account when calculating $T_{3,q}^{N,\text{ns}}$. In eq.(44) one has to substitute on the left hand side for $T_{3,q}^{N,\text{ns}}$

$$\left(T_{3,q}^{N,\text{ns}}\right)^{\text{bare}} = (Z_5 Z_A)^{-1} \left(T_{3,q}^{N,\text{ns}}\right)^{\text{ren}}. \quad (54)$$

This concludes our review of the OPE formalism to calculate Mellin moments of DIS structure functions.

3 Methods

In this section we will discuss the methods used to obtain the Mellin moments of the DIS structure functions.

The idea is to determine reduction identities based on sets of derivative equations for the N -th Mellin moment of a given diagram in dimensional regularization [27, 28]. This is done such, that it is possible to set up systematically recursion relations in the Mellin moment N . Solving these recursions leads to multiple nested sums, which successively can be expressed in terms of harmonic sums, the basic functions in Mellin space. This method of recursions was previously used in ref.[10] and indeed, a number of the relations we give can also be found there.

3.1 Reduction identities

Let us begin with a classification of the relevant topologies following the notations of refs.[36, 37]. We will begin with 2-point functions of external momentum q and subsequently dress them up by inserting two additional external legs with momentum p .

The basic topologies of a 2-point function up to two loops are shown below. They are given by the basic 1-loop diagram $L1$ and the 2-loop topology named $T1$, where the off-shell q -momentum flows from right to left through the diagrams.



Figure 1: The basic basic topologies of a 2-point function up to two loops. Left: The basic 1-loop diagram $L1$. Right: The 2-loop topology $T1$.

The other 2-loop topologies denoted $T2$ and $T3$ in refs.[36, 37] are just special cases of $T1$, where one of the lines 1, ..., 5 is missing. They reduce to the convolution or to the product of two 1-loop integrals.

For the calculation of the N -th Mellin moment of deep inelastic structure functions, we need to consider 4-point diagrams with external momenta p , $p^2 = 0$, and q . Then, all topologies up to two loops that contribute can be constructed from the diagrams in fig.1 by attaching two p -dependent external legs in all topologically independent ways to the various lines. We call these composite topologies. For the calculation we need to construct only a single set of programs that can deal with the N -th Mellin moment of the various composite topologies.

Because the method relies on expressing composite topologies into simpler composite topologies we introduce also the notion of basic building blocks. These are composite topologies in which both p -lines attach to the same line in the basic topology. At the two loop level there are two basic building blocks:

$$T_{111} = \text{Diagram} = \int d^D p_1 d^D p_2 \frac{1}{(p_1^2)^a ((p+p_1)^2)^A (p_2^2)^b (p_3^2)^c (p_4^2)^d (p_5^2)^e}, \quad (55)$$

$$T_{155} = \text{Diagram} = \int d^D p_1 d^D p_2 \frac{1}{(p_1^2)^a (p_2^2)^b (p_3^2)^c (p_4^2)^d (p_5^2)^e ((p+p_5)^2)^E}. \quad (56)$$

In the pictorial representation that we use for the diagrams we indicate the p -flow through the diagrams by fat lines. The numbers indicate the number of powers of denominators in the indicated momentum. If the same line has two numbers, the second one indicates the number of powers of $2p \cdot p_i$ in which i is the number of the line in the corresponding basic topology. Hence

$$\text{Diagram} = \int d^D p_1 d^D p_2 \frac{(2p \cdot p_1)^m (2p \cdot p_5)^n}{(p_1^2)^a (p_2^2)^b (p_3^2)^c (p_4^2)^d (p_5^2)^e}. \quad (57)$$

We will need the composite topologies $T_{111}, T_{112}, T_{113}, T_{114}, T_{115}, T_{116}, T_{117}, T_{155}, T_{156}$ and T_{167} . All others are related to these by symmetry operations.

The first observation is that when the external legs 6 and/or 7 are involved, one can expand the propagator(s) in $p + q$ with the formula

$$\frac{1}{(p+q)^{2n}} = \sum_{i=0}^{\infty} \frac{1}{(q^2)^n} (-1)^i \binom{n+i-1}{i} \frac{(2p \cdot q)^i}{(q^2)^i}. \quad (58)$$

If we use n powers of $2p \cdot q$ from this expansion, we need only $N - n$ powers of $2p \cdot q$ from the rest of the diagram. Hence such a composite topology becomes a single sum over a simpler topology in which p flows through one line fewer. This solves 4 of the 10 composite two loop topologies.

The way to deal with the other 6 composite topologies is to use all different variations of the integration by parts identities and some scaling identities. The integration by parts identities [27, 38] are of the type

$$\int d^D p_1 d^D p_2 \frac{\partial}{\partial p_i^\mu} p_j^\mu T_{1kl} = 0, \quad (59)$$

because we deal with a total derivative here. The momenta p_i and p_j can be equal to any of the internal momenta. Additionally p_j can be equal to p or q . The scaling identities involve applying one of the operators

$$q^\mu \frac{\partial}{\partial q^\mu}, \quad p^\mu \frac{\partial}{\partial q^\mu}, \quad p^\mu \frac{\partial}{\partial p^\mu}. \quad (60)$$

both inside the integral and to the integrated result in Mellin space. The fourth operator of this kind cannot be used, because it leads to an inequality when applied to $p^2 = 0$. Sometimes a third type of identities can be useful. It is along the lines of the Passarino–Veltman decomposition in formfactors [39]:

$$\int d^D p_1 d^D p_2 p_i^\mu T_{1kl} = q^\mu I_{kl}^{(q)} + p^\mu I_{kl}^{(p)}. \quad (61)$$

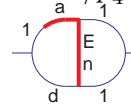
First the two formfactors $I^{(q)}$ and $I^{(p)}$ are determined by contracting eq.(61) either with q^μ or p^μ . Then the relevant identity can be obtained by taking the derivative with respect to q^μ . This can be

using integration by parts in the left triangle of the diagram. The final result is

$$\begin{aligned}
(\tilde{N} + 2) \frac{D-5}{2} \text{Diagram}_1 &= 2 \text{Diagram}_2 - 2 \text{Diagram}_3 \\
&+ \text{Diagram}_4 - \text{Diagram}_5,
\end{aligned} \tag{65}$$

and we see three trivial integrals and one integral of the type $T1_{11}$, albeit with different coefficients after the expansion in p .

For the $T1_{14}$ topology we have to consider two steps. The first is to reduce the integrals containing $1/(p_1^2(p+p_1)^2 p_4^2(p+p_4)^2)$. This is trivial, because we can use the scaling identity based on $p^\mu \partial/\partial p^\mu$ which removes either the $1/p_1^2$ or the $1/p_4^2$. After that we can reroute the momentum p and we are

left with a diagram of the type  in which E is normally 1 and d is 2. We have to be

careful with the signs if the diagram has to be turned upside down. We are now left with basically one diagram with the p momentum flowing through the lines 1 and 5.

Further reduction can happen in a variety of ways. The important thing to pay attention to is that during the reduction, one does not generate simpler integrals with an additional factor of $1/\epsilon$. In that case these simpler integrals would have to be expanded to higher powers in ϵ which may lead to relatively difficult sums. Hence we follow a rather elaborate scheme that is free of these problems by combining all the equations we can write down for the diagram. It results in:

$$\begin{aligned}
\text{Diagram}_1 \cdot q \cdot q &= (\tilde{N} + E - n - D + 5) \text{Diagram}_2 + n \text{Diagram}_3 \\
&+ \text{Diagram}_4 + E \text{Diagram}_5 - n \text{Diagram}_6 - E \text{Diagram}_7,
\end{aligned} \tag{66}$$

$$\begin{aligned}
\text{Diagram}_1 \cdot q \cdot q &= (E - n - D + 3) \text{Diagram}_2 - n \text{Diagram}_3 \\
&- E \text{Diagram}_4 + n \text{Diagram}_5 + E \text{Diagram}_6 + \text{Diagram}_7,
\end{aligned} \tag{67}$$

$$\begin{aligned}
\text{Diagram}_1 &= \frac{1}{\tilde{N} + 1 - n} \left(\text{Diagram}_2 + \text{Diagram}_3 - \text{Diagram}_4 \right. \\
&+ \frac{2p \cdot q}{q \cdot q} \left((D - 4 - \tilde{N} - E + n) \text{Diagram}_5 + E \text{Diagram}_6 - E \text{Diagram}_7 \right. \\
&\left. \left. - \text{Diagram}_8 + n \text{Diagram}_9 - n \text{Diagram}_{10} \right) \right).
\end{aligned} \tag{68}$$

The second of these equations we will need when we treat the $T1_{13}$ topology. The third equation leads to a recursion which for the fourth term in the right hand side is terminated when enough powers of $2p \cdot q/q^2$ have been pulled out. All other terms in the right hand side are of a simpler nature. There is no ϵ in any denominator.

scaling operator $p^\mu \partial / \partial p^\mu$, after which the p -flow is restored to how it was in the beginning we obtain the relation

$$(N+a+b) \text{diagram} = a \text{diagram} + b \text{diagram} - (N-1+n) \frac{2p \cdot q}{q \cdot q} \text{diagram} . \quad (73)$$

It is clear that in the case of A and B being one, the number of lines that contain the momentum p will diminish by one. Hence this relation will be very useful in the future. In the special case of $T1_{13}$ in eq.(71) the first two terms in the right hand side are identical and $n = 7 - D$.

Many of these equations involve recursion relations and their solution can be written down as a single sum over the simpler diagram(s). Each of these sums is a so-called single parameter sum. With this we mean that the summand after expansion in ϵ is only a function of the summation parameter and the upper limit of the summation, the lower limit being either zero or one. In this hides the power of the method, because this avoids multiple sums that cannot be done with current techniques.

3.2 Solutions

The equations for the various composite topologies are mostly recursion relations, or in another terminology: first order difference equations. Suppose we have the equation

$$a(N)F(N) - b(N)F(N-1) - G(N) = 0, \quad (74)$$

then its solution will be

$$F(N) = \frac{\prod_{j=1}^N b(j)}{\prod_{j=1}^N a(j)} F(0) + \sum_{i=1}^N \frac{\prod_{j=i+1}^N b(j)}{\prod_{j=i}^N a(j)} G(i). \quad (75)$$

In the case that the functions $a(N)$ and $b(N)$ can be factorized in linear polynomials in N with the coefficient of N being one and the other coefficients being integers, the products can be written as combinations of Γ -functions. Because a and b may also depend on the parameter ϵ the Γ -functions should be expanded around $\epsilon = 0$. This will lead to factorials and harmonic sums. The diagram indicated by $F(0)$ in eq.(75) can be evaluated with standard MINCER techniques [36, 37]. If the function $G(N)$ is expressed as a power series in ϵ with the coefficients being combinations of harmonic sums in $N + m$ and powers of $N + m$, m being a fixed integer, the sum in eq.(75) can be done and $F(N)$ will be a combination of harmonic sums in $N + k$ and powers of $N + k$ with k being a fixed integer.

Eventually, all solutions to the recursion relations, which were derived above, can be written in terms of harmonic sums. Thus, before we continue we will give some attention to these functions that describe the solutions in Mellin space.

3.3 Harmonic sums

Harmonic sums are the basic functions in Mellin space and much about them can be found in refs.[4, 22, 23]. We follow here the notations and the definitions of ref.[22]. Harmonic sums are defined by

$$S_m(N) = \sum_{i=1}^N \frac{1}{i^m}, \quad S_{-m}(N) = \sum_{i=1}^N \frac{(-1)^m}{i^m}, \quad (76)$$

while higher functions can be defined recursively

$$S_{m_1, \dots, m_k}(N) = \sum_{i=1}^N \frac{1}{i^{m_1}} S_{m_2, \dots, m_k}(i), \quad (77)$$

$$S_{-m_1, \dots, m_k}(N) = \sum_{i=1}^N \frac{(-1)^{m_1}}{i^{m_1}} S_{m_2, \dots, m_k}(i). \quad (78)$$

These functions appear, among others, when Γ -functions are expanded in terms of ϵ . But they also show up in sums of the type

$$\sum_{i=1}^n (-1)^i \binom{n}{i} \frac{1}{n^3} = -S_{1,1,1}(n), \quad (79)$$

which are rather common in our calculations.

The weight of a sum is defined as the sum of the absolute values of all the indices m_i . These sums form an algebra. Hence the product of two sums with the same argument can be written as a sum of terms, each with a single sum of which the weight is the sum of the weights of the original two sums. Many sums involving these harmonic sums can be done by automatic algorithms to any level of complexity. For us the important sums are

$$\begin{aligned} & \sum_{i=1}^n \frac{S_{\vec{p}}(i) S_{\vec{q}}(n-i)}{i^m}, \\ & \sum_{i=1}^n (-1)^i \frac{S_{\vec{p}}(i) S_{\vec{q}}(n-i)}{i^m}, \\ & \sum_{i=1}^n (-1)^i \binom{n}{i} \frac{S_{\vec{p}}(i)}{i^m}, \\ & \sum_{i=1}^n (-1)^i \binom{n}{i} \frac{S_{\vec{p}}(i) S_{\vec{q}}(n-i)}{i^m}, \end{aligned}$$

in which we interpret an S without indices as 1 and $\vec{p} = p_1, \dots, p_v$; $\vec{q} = q_1, \dots, q_w$. The evaluation of these sums is described in [22] and has been programmed in the language of FORM [41]. The important thing to note is that these sums all evaluate into harmonic sums and denominators of the same argument as the harmonic sums. In practise this means that with the sums we run into we will remain inside the space of harmonic sums and hence the harmonic sums will span the solution space of our integrals in Mellin space.

3.4 Harmonic Polylogarithms

In x -space the results will be presented in terms of harmonic polylogarithms [24]. These functions are related to the multi-dimensional polylogarithms of ref.[42]. They form a natural basis in the space of inverse Mellin transformations of harmonic sums. We will mention here their most important properties. For more details the reader should consult ref.[24].

The harmonic polylogarithms of weight w and with argument x are identified by a set of m_1, \dots, m_w indices which can take one of the three values 0, 1, -1. Harmonic polylogarithms are denoted by $H_{m_1, \dots, m_w}(x)$ and explicitly, for lowest weight one defines

$$H_0(x) = \ln x, \quad (80)$$

$$H_1(x) = \int_0^x \frac{dx'}{1-x'} = -\ln(1-x), \quad (81)$$

$$H_{-1}(x) = \int_0^x \frac{dx'}{1+x'} = \ln(1+x). \quad (82)$$

For their derivatives, one has

$$\frac{d}{dx} H_m(x) = f_m(x) , \quad (83)$$

where again the index m can take the values $0, +1, -1$ and the three rational fractions $f_m(x)$ are given by

$$f_0(x) = \frac{1}{x}, \quad f_1(x) = \frac{1}{1-x}, \quad f_{-1}(x) = \frac{1}{1+x}. \quad (84)$$

In general, the harmonic polylogarithms of weight w are then defined as follows,

$$H_{m_1, \dots, m_w}(x) = \frac{1}{w!} \ln^w x, \quad \text{if } m_1, \dots, m_w = 0, \dots, 0, \quad (85)$$

while, if $m_1, \dots, m_w \neq 0, \dots, 0$

$$H_{m_1, \dots, m_w}(x) = \int_0^x dz f_{m_1}(z) H_{m_2, \dots, m_w}(z). \quad (86)$$

To provide a link to the standard literature, we give the results for harmonic polylogarithms up to weight three in terms of common polylogarithms in the appendix A.

Just like the harmonic sums the H -functions form an algebra. This is to say: products of two H -functions, $H_{\vec{m}_w}(x) H_{\vec{n}_v}(x)$ of weight w and v respectively, where $\vec{m}_w = m_1, \dots, m_w$ and $\vec{n}_v = n_1, \dots, n_v$ can be expressed in a sum of H -function of weight $w + v$,

$$H_{\vec{m}_w}(x) H_{\vec{n}_v}(x) = \sum_{\vec{l}_{w+v} = \vec{m}_w \uplus \vec{n}_v} H_{\vec{l}_{w+v}}(x), \quad (87)$$

in which $\vec{m}_w \uplus \vec{n}_v$ represents all mergers of \vec{m}_w and \vec{n}_v in which the relative orders of the elements of \vec{m}_w and \vec{n}_v are preserved. Hence the sum consists of $(v+w)!/v!w!$ terms. This can be shown by induction using integration by parts and eq.(86). It should be realized that the rules of this algebra are complementary to those of the sums in the sense that the algebra of H -functions in x is related to the extra algebraic rules for S -sums in infinity, while the extra algebraic rules for H -functions in 1 are related to the algebraic rules for S -sums in N .

The harmonic polylogarithms may be expanded in a Taylor series with the expansion coefficients being harmonic sums,

$$H_{\vec{m}}(x) = \sum_{i=1}^{\infty} \sum_{\vec{n}} c_{\vec{m}\vec{n}} \frac{\sigma_{\vec{n}}^i x^i}{i} S_{\vec{n}}(i), \quad (88)$$

where $\sigma = \pm 1$ and $S_{\vec{n}}$ is a harmonic sum of weight v , $\vec{n} = n_1, \dots, n_w$. In general eq.(88) only holds, if $\vec{m} = m_1, \dots, m_w$ has no trailing zeroes in the index field. Those correspond to factors $\ln(x)$, which do not admit a regular Taylor expansion. However, trailing zeroes in the index field can be factored out by repeated use of the product identity eq.(87), such that eq.(88) can safely be applied to the left-over H -function, which by construction does not contain powers of $\ln(x)$ anymore.

Finally, we define the Mellin transformation of regular functions as

$$f(N) = \int_0^1 dx x^{N-1} f(x). \quad (89)$$

The Mellin transformation of $1/(1-x)$ and possible powers of logarithms $\ln(1-x)$ are regularized in the sense of $+$ -distributions. For this we have to extract first the powers of $\ln(1-x)$ which correspond to leading indices with the value 1. The extraction is similar to the extraction of trailing zeroes. Hence:

$$\int_0^1 dx x^{N-1} \left[\frac{H_{1, \dots, 1}(x) H_{\vec{m}}(x)}{1-x} \right]_+ = \int_0^1 dx \left(x^{N-1} H_{\vec{m}}(x) - H_{\vec{m}}(1) \right) \frac{H_{1, \dots, 1}(x)}{1-x}, \quad (90)$$

in which \vec{m} does not have a 1 for its first element.

Together the above relations allow the construction of the Mellin transform of any H -function. It turns out that there is a one to one relationship between functions of the type $H_{\vec{m}}(x)/(1 \pm x)$ with \vec{m} having weight w and S -sums of weight $w + 1$. Hence it is possible to construct the inverse Mellin transform of any result in N space that can be expressed in terms of harmonic sums. The complete algorithm has been coded in the language of FORM [41].

4 Calculation and Results

In this section, we will discuss our calculation of the DIS structure functions and list our results obtained for the complete set of anomalous dimensions, the flavour non-singlet and singlet quark and the gluon coefficient functions up to two loops.

However, first a few remarks on details are in order. All Feynman diagrams contributing to the structure functions have been generated with the help of QGRAF [43] and have been stored in a database. For the calculation of the even Mellin moments of the partonic invariants $T_{2,p}^N$ and $T_{L,p}^N$ we used a database identical with the one in ref.[19], which contains 425 diagrams up to two loops. To obtain the full information about $T_{3,p}^N$ and the complete non-singlet contribution to $T_{2,p}^N$, we have generated a new database of 360 diagrams corresponding to the four different structure functions $F_2^{\nu P \pm \bar{\nu} P}$ and $F_3^{\nu P \pm \bar{\nu} P}$.

Both databases have been checked by calculating some lower fixed Mellin moments in an arbitrary covariant gauge with the MINCER algorithm [37], keeping the gauge parameter ξ in the gluon propagator, that is to say,

$$i \frac{-g^{\mu\nu} + (1 - \xi)q^\mu q^\nu}{q^2 - i\epsilon}, \quad (91)$$

and in the final result all dependence on ξ does cancel.

The calculation of the partonic invariants $T_{i,p}^N$, $i = 2, 3, L$ and $T_{\phi,p}^N$ requires to consider only external quarks and gluons with physical polarization. In the case of external quarks, the sum over all polarizations leads to the projection operator \not{p} that closes the open string of Dirac matrices associated with the external quark line. For external gluons on the other hand, the sum over all physical polarizations can be done by contracting the external gluon lines with

$$-g^{\alpha\beta} + \frac{p^\alpha q^\beta + p^\beta q^\alpha}{p \cdot q} - \frac{p^\alpha p^\beta q^2}{(p \cdot q)^2}, \quad (92)$$

where p is on-shell, $p^2 = 0$. An alternative approach contracts the external gluon lines only with $-g^{\alpha\beta}$. To remove the unphysical contributions, one adds extra classes of diagrams, with external ghosts instead of external gluons. The latter approach has been used in ref.[19] while we have checked that both methods agree for our calculation of the N -th Mellin moment up to two loops.

The explicit calculation of the DIS structure functions has been done with the symbolic manipulation program FORM [41]. To that end, all recursion relations given in section 3 have been implemented in a program, that reduces the Feynman diagrams of DIS structure functions up to two loops to multiple nested sums over the basic building blocks. Subsequently the program calls the SUMMER algorithm [22], to solve these nested sums in terms of the basis of harmonic sums. To speed up the calculation the results for a large number of basic integrals have been tabulated.

Let us emphasize that the method of recursion relations as described in section 3 allows for numerous checks at all stages of the calculation by means of the standard MINCER routine [37]. This is from a practical view point by far the most powerful feature of our approach, because the debugging of all our programs is extremely simplified.

A final remark is concerned with the analytic continuation of our N -space results. In the framework of the OPE we calculate either the even or the odd Mellin moments for a given structure function

depending on the operators that contribute as detailed in section 2. This is sufficient to determine all moments in the complex N plane by analytic continuation. In order to do so we first perform the inverse Mellin transformation of our N -space result to obtain the complete expression in x -space. This mapping is unique provided we fix all factors of $(-1)^N$ according to whether we started from even or odd Mellin moments. Subsequently, we may take this x -space result and execute another Mellin transformation back to N -space to obtain the final answer in terms of harmonic sums valid for all non-negative integer Mellin moments. This step of restoring all those factors of $(-1)^N$ which are due to analyticity is particularly important for the determination of the flavour non-singlet anomalous dimensions and coefficient functions at two loops as will be detailed below.

4.1 Renormalization and mass factorization

Let us now concentrate on the issue of renormalization and mass factorization for the Mellin moments of the DIS structure functions. We wish to show explicitly how to extract the anomalous dimensions and coefficient functions from eqs.(44), (45) and (47) if the partonic invariants $T_{i,p}^N$ and $T_{\phi,p}^N$ are expanded in powers of α_s and ϵ . In particular, we will obtain the 2-loop anomalous dimensions $\gamma_{gq}^{(1)}$ and $\gamma_{gg}^{(1)}$ from the invariants $T_{\phi,p}^N$ of the scalar particle.

The calculation is performed in dimensional regularization $D = 4 - 2\epsilon$, [27, 28]. We briefly recall the procedure outlined in section 2 and in refs.[14, 16, 19]. The sum of all Feynman diagrams contributing to a given partonic invariant $T_{i,p}^N$ or $T_{\phi,p}^N$ contains only ultraviolet and collinear divergences. The ultraviolet divergences need to be removed by coupling constant renormalization changing the bare α_s^{bare} to the renormalized α_s as defined in eq.(36). If necessary, the additional renormalizations associated with the axial current, γ_5 and the interaction term $\phi F_{\mu\nu}^a F_a^{\mu\nu}$ of the scalar particle ϕ have to be taken into account as described in section 2 and given in eqs.(49) and (54).

Then one is left with the collinear divergences associated with the partonic initial states. Those need to be removed by mass factorization changing the bare densities of partons in the hadron to renormalized ones. This is the same as renormalizing the operator matrix elements $A_{p,N}^j$ as done in eqs.(44), (45) and (47).

In the following, we list the results for the Mellin moments of the parton invariants order by order in α_s . To that end it is useful to define the following expansions in powers of α_s for the coefficient functions and the anomalous dimensions,

$$\gamma_{pp}(\alpha_s) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi} \right)^{n+1} \gamma_{pp}^{(n)}, \quad (93)$$

$$C_{i,p}(\alpha_s) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi} \right)^n c_{i,p}^{(n)}, \quad (94)$$

and, since the operator matrix elements $A_{p,N}^{\text{p,tree}}$ factorize after application of the projector \mathcal{P}_N , also for the parton invariants,

$$T_{i,p}^N = \left(T_{i,p}^{(0)} + T_{i,p}^{(1)} + T_{i,p}^{(2)} + \dots \right) A_{p,N}^{\text{p,tree}}, \quad (95)$$

where all left-over singularities in the $T_{i,p}^{(l)}$ are of collinear nature. The procedure of mass factorization works iteratively order by order, both in α_s and in the regularization parameter ϵ , such that all physical quantities can be uniquely extracted. Notice in particular, that the gluon tree level operator matrix element $A_{g,N}^{\text{g,tree}}$ is $(1 - \epsilon)\text{const}_N$. If factorized as in eq.(95), the factor $(1 - \epsilon)$ due to the gluon polarization in D dimensions is essential for a proper determination of the 1-loop coefficient function $c_{2,g}^{(1)}$. As discussed in ref.[16], the omission of the factor $(1 - \epsilon)$ accounts for some of the differences with ref.[10] for the structure function F_L .¹

¹ The discrepancies were corrected in ref.[44].

All results are given in dimensional regularization in the $\overline{\text{MS}}$ -scheme with α_s being the renormalized quantity according to eq.(36). At leading order, we have normalized,

$$T_{2,q}^{(0)} = T_{3,q}^{(0)} = 1, \quad T_{2,g}^{(0)} = T_{L,q}^{(0)} = T_{L,g}^{(0)} = 0. \quad (96)$$

At first order in α_s , we have to expand up to order ϵ and find,

$$T_{2,q}^{(1)} = \frac{\alpha_s}{4\pi} S_\epsilon \left(\frac{\mu^2}{Q^2} \right)^\epsilon \left[\frac{1}{\epsilon} \gamma_{qq}^{(0)} + c_{2,q}^{(1)} + \epsilon a_{2,q}^{(1)} \right], \quad (97)$$

$$T_{3,q}^{(1)} = \frac{\alpha_s}{4\pi} S_\epsilon \left(\frac{\mu^2}{Q^2} \right)^\epsilon \left[\frac{1}{\epsilon} \gamma_{qq}^{(0)} + c_{3,q}^{(1)} + \epsilon a_{3,q}^{(1)} \right], \quad (98)$$

$$T_{2,g}^{(1)} = n_f \frac{\alpha_s}{4\pi} S_\epsilon \left(\frac{\mu^2}{Q^2} \right)^\epsilon \left[\frac{1}{\epsilon} \gamma_{qg}^{(0)} + c_{2,g}^{(1)} + \epsilon a_{2,g}^{(1)} \right], \quad (99)$$

$$T_{L,q}^{(1)} = \frac{\alpha_s}{4\pi} S_\epsilon \left(\frac{\mu^2}{Q^2} \right)^\epsilon \left[c_{L,q}^{(1)} + \epsilon a_{L,q}^{(1)} \right], \quad (100)$$

$$T_{L,g}^{(1)} = n_f \frac{\alpha_s}{4\pi} S_\epsilon \left(\frac{\mu^2}{Q^2} \right)^\epsilon \left[c_{L,g}^{(1)} + \epsilon a_{L,g}^{(1)} \right], \quad (101)$$

where the factor S_ϵ is defined by

$$S_\epsilon = \exp(\epsilon \{\ln(4\pi) - \gamma_E\}). \quad (102)$$

At second order in α_s , we need to split up the contributions into flavour non-singlet and singlet parts. Allowing for electroweak interactions, one can consider in the non-singlet case the structure functions of four different physical processes, $F_2^{\nu P \pm \overline{\nu} P}$ and $F_3^{\nu P \pm \overline{\nu} P}$. This implies that we also need to distinguish even and odd moments. We have

$$\begin{aligned} T_{2,q}^{(2),\text{ns},\pm} &= \left(\frac{\alpha_s}{4\pi} \right)^2 S_\epsilon^2 \left(\frac{\mu^2}{Q^2} \right)^{2\epsilon} \left[\frac{1}{\epsilon^2} \left\{ \frac{1}{2} \left(\gamma_{qq}^{(0)} \right)^2 - \frac{1}{2} \beta_0 \gamma_{qq}^{(0)} \right\} \right. \\ &\quad \left. + \frac{1}{\epsilon} \left\{ \frac{1}{2} \gamma_{qq}^{(1),\pm,V} + \gamma_{qq}^{(0)} c_{2,q}^{(1)} \right\} + c_{2,q}^{(2),\text{ns},+} \pm c_{2,q}^{(2),\text{ns},-} + \gamma_{qq}^{(0)} a_{2,q}^{(1)} \right], \end{aligned} \quad (103)$$

$$\begin{aligned} T_{3,q}^{(2),\text{ns},\pm} &= \left(\frac{\alpha_s}{4\pi} \right)^2 S_\epsilon^2 \left(\frac{\mu^2}{Q^2} \right)^{2\epsilon} \left[\frac{1}{\epsilon^2} \left\{ \frac{1}{2} \left(\gamma_{qq}^{(0)} \right)^2 - \frac{1}{2} \beta_0 \gamma_{qq}^{(0)} \right\} \right. \\ &\quad \left. + \frac{1}{\epsilon} \left\{ \frac{1}{2} \gamma_{qq}^{(1),\pm,V} + \gamma_{qq}^{(0)} c_{3,q}^{(1)} \right\} + c_{3,q}^{(2),\text{ns},+} \pm c_{3,q}^{(2),\text{ns},-} + \gamma_{qq}^{(0)} a_{3,q}^{(1)} \right], \end{aligned} \quad (104)$$

$$T_{L,q}^{(2),\text{ns}} = \left(\frac{\alpha_s}{4\pi} \right)^2 S_\epsilon^2 \left(\frac{\mu^2}{Q^2} \right)^{2\epsilon} \left[\frac{1}{\epsilon} \left\{ \gamma_{qq}^{(0)} c_{L,q}^{(1)} \right\} + c_{L,q}^{(2),\text{ns}} + \gamma_{qq}^{(0)} a_{L,q}^{(1)} \right]. \quad (105)$$

where the \pm -sign denotes even or odd moments in the expressions for $T_{2,q}^{(2),\text{ns},\pm}$ and $T_{3,q}^{(2),\text{ns},\pm}$. We want to emphasize this distinction for even and odd moments as done in eqs.(103) and (104) is due to the fact that we deal with different scattering processes.

It is relevant for the 2-loop anomalous dimensions entering in eqs.(103) and (104) because of flavour symmetry breaking in the anti-quark distributions. We use the notation of eq.(32)

$$\gamma_{qq}^{(1),\pm,V} = \gamma_{qq}^{(1),V} \pm \gamma_{q\overline{q}}^{(1),V}. \quad (106)$$

It is also relevant for the 2-loop coefficient functions, which decompose for even and odd moments as

$$c_{2,q}^{(2),\text{ns}} = c_{2,q}^{(2),\text{ns},+} + (-1)^N c_{2,q}^{(2),\text{ns},-}, \quad (107)$$

$$c_{3,q}^{(2),\text{ns}} = c_{2,q}^{(2),\text{ns},+} + (-1)^N c_{3,q}^{(2),\text{ns},-}. \quad (108)$$

After analytical continuation of our N -space results we can reconstruct these physical signs $(-1)^N$ in eqs.(106)–(108). That is to say, we can then determine γ_{qq}^V and $\gamma_{q\bar{q}}^V$ by taking the sum or the difference of $\gamma_{qq}^{+,V}$ and $\gamma_{q\bar{q}}^{-,V}$. Analogously, the individual contributions to the non-singlet coefficient functions at two loops, $c_{2,q}^{\text{ns},+}$, $c_{2,q}^{\text{ns},-}$, $c_{3,q}^{\text{ns},+}$, and $c_{3,q}^{\text{ns},-}$, are obtained, but again the reconstruction works only after analytical continuation.

Note also, that in the equations above the partonic invariants $T_{3,q}^{(1)}$ and $T_{3,q}^{(2),\text{ns}}$ are always understood to be renormalized according to eq.(54).

We decompose the quark singlet invariant $T_{i,q}^{(2),s}$ into non-singlet and pure-singlet contributions,

$$T_{2,q}^{(2),s} = T_{2,q}^{(2),\text{ns},+} + T_{2,q}^{(2),\text{ps}}, \quad T_{L,q}^{(2),s} = T_{L,q}^{(2),\text{ns}} + T_{L,q}^{(2),\text{ps}}, \quad (109)$$

and for the pure-singlet contributions we have at second order in α_s ,

$$T_{2,q}^{(2),\text{ps}} = n_f \left(\frac{\alpha_s}{4\pi} \right)^2 S_\epsilon^2 \left(\frac{\mu^2}{Q^2} \right)^{2\epsilon} \left[\frac{1}{\epsilon^2} \left\{ \frac{1}{2} \gamma_{qg}^{(0)} \gamma_{gq}^{(0)} \right\} + \frac{1}{\epsilon} \left\{ \frac{1}{2} \gamma_{qq}^{(1),+,S} + \gamma_{gq}^{(0)} c_{2,g}^{(1)} \right\} + c_{2,q}^{(2),\text{ps}} + \gamma_{gq}^{(0)} a_{2,g}^{(1)} \right], \quad (110)$$

$$T_{L,q}^{(2),\text{ps}} = n_f \left(\frac{\alpha_s}{4\pi} \right)^2 S_\epsilon^2 \left(\frac{\mu^2}{Q^2} \right)^{2\epsilon} \left[\frac{1}{\epsilon} \left\{ \gamma_{gq}^{(0)} c_{L,g}^{(1)} \right\} + c_{L,q}^{(2),\text{ps}} + \gamma_{gq}^{(0)} a_{L,g}^{(1)} \right]. \quad (111)$$

In eq.(110) we used again the notation for the anomalous dimensions of eqs.(33) and (34)

$$\gamma_{qq}^{(1),\pm,S} = \gamma_{qq}^{(1),S} \pm \gamma_{q\bar{q}}^{(1),S}. \quad (112)$$

The pure singlet contribution to $T_{3,q}^{(2),\text{ps}}$ vanishes, implying that we have

$$\gamma_{qq}^{(1),-,S} = \gamma_{qq}^{(1),S} - \gamma_{q\bar{q}}^{(1),S} = 0. \quad (113)$$

For the gluonic invariants, we find at second order in α_s ,

$$T_{2,g}^{(2)} = n_f \left(\frac{\alpha_s}{4\pi} \right)^2 S_\epsilon^2 \left(\frac{\mu^2}{Q^2} \right)^{2\epsilon} \left[\frac{1}{\epsilon^2} \left\{ \frac{1}{2} \gamma_{qg}^{(0)} (\gamma_{qq}^{(0)} + \gamma_{gg}^{(0)}) - \frac{1}{2} \beta_0 \gamma_{qg}^{(0)} \right\} + \frac{1}{\epsilon} \left\{ \frac{1}{2} \gamma_{qg}^{(1)} + \gamma_{gg}^{(0)} c_{2,g}^{(1)} + \gamma_{qg}^{(0)} c_{2,q}^{(1)} \right\} + c_{2,g}^{(2)} + \gamma_{gg}^{(0)} a_{2,g}^{(1)} + \gamma_{qg}^{(0)} a_{2,q}^{(1)} \right], \quad (114)$$

$$T_{L,g}^{(2)} = n_f \left(\frac{\alpha_s}{4\pi} \right)^2 S_\epsilon^2 \left(\frac{\mu^2}{Q^2} \right)^{2\epsilon} \left[\frac{1}{\epsilon} \left\{ \gamma_{gg}^{(0)} c_{L,g}^{(1)} + \gamma_{qg}^{(0)} c_{L,q}^{(1)} \right\} + c_{L,g}^{(2)} + \gamma_{gg}^{(0)} a_{L,g}^{(1)} + \gamma_{qg}^{(0)} a_{L,q}^{(1)} \right]. \quad (115)$$

Finally, we can give the expressions for the partonic invariants of the scalar particle ϕ . We find at leading order,

$$T_{\phi,q}^{(0)} = 0, \quad T_{\phi,g}^{(0)} = 1. \quad (116)$$

In the equations below the partonic invariants $T_{\phi,p}^{(1)}$ and $T_{\phi,p}^{(2)}$ are always understood to be renormalized according to eq.(49). Then we obtain at first order in α_s ,

$$T_{\phi,q}^{(1)} = \frac{\alpha_s}{4\pi} S_\epsilon \left(\frac{\mu^2}{Q^2} \right)^\epsilon \left[\frac{1}{\epsilon} \gamma_{gq}^{(0)} + c_{\phi,q}^{(1)} \right], \quad (117)$$

$$T_{\phi,g}^{(1)} = \frac{\alpha_s}{4\pi} S_\epsilon \left(\frac{\mu^2}{Q^2} \right)^\epsilon \left[\frac{1}{\epsilon} \gamma_{gg}^{(0)} + c_{\phi,g}^{(1)} \right], \quad (118)$$

where we only have to expand up the finite terms in ϵ .

Finally, at second order in α_s ,

$$T_{\phi,q}^{(2)} = \left(\frac{\alpha_s}{4\pi} \right)^2 S_\epsilon^2 \left(\frac{\mu^2}{Q^2} \right)^{2\epsilon} \left[\frac{1}{\epsilon^2} \left\{ \frac{1}{2} \gamma_{gq}^{(0)} (\gamma_{gq}^{(0)} + \gamma_{gg}^{(0)}) - \frac{1}{2} \beta_0 \gamma_{gq}^{(0)} \right\} + \frac{1}{\epsilon} \left\{ \frac{1}{2} \gamma_{gq}^{(1)} + \gamma_{gq}^{(0)} c_{\phi,g}^{(1)} + \gamma_{gq}^{(0)} c_{\phi,q}^{(1)} \right\} \right], \quad (119)$$

$$T_{\phi,g}^{(2)} = \left(\frac{\alpha_s}{4\pi} \right)^2 S_\epsilon^2 \left(\frac{\mu^2}{Q^2} \right)^{2\epsilon} \left[\frac{1}{\epsilon^2} \left\{ \frac{1}{2} (\gamma_{gg}^{(0)})^2 + \frac{1}{2} \gamma_{qg}^{(0)} \gamma_{gq}^{(0)} - \frac{1}{2} \beta_0 \gamma_{gg}^{(0)} \right\} + \frac{1}{\epsilon} \left\{ \frac{1}{2} \gamma_{gg}^{(1)} + \gamma_{gg}^{(0)} c_{\phi,g}^{(1)} + \gamma_{qg}^{(0)} c_{\phi,q}^{(1)} \right\} \right]. \quad (120)$$

This concludes our brief discussions on the extraction of the anomalous dimensions and coefficient functions from eqs.(44), (45) and (47) by means of eqs.(96)–(120).

4.2 Results in Mellin space

We are now ready to present our results, which provide all necessary ingredients for the solution of the renormalization group equations (39) and (40) for the flavour singlet and non-singlet coefficient functions as detailed for example in ref.[45]. However, as is well known, a complete solution to NNLO for the scale evolution of the coefficient functions requires also the still unknown anomalous dimensions $\gamma_{pp}^{(2)}$ at 3-loops.

To summarize, our results for the anomalous dimensions agree with the ones published in refs.[1, 3, 4, 5, 6, 7, 8] while our results for the coefficient functions agree with those of refs.[2, 13, 14, 15, 16]. As far as the earlier calculations of the longitudinal structure function F_L at two loops of refs.[9, 10, 11, 12] are concerned, let us mention that there is only complete agreement for $c_{L,q}^{NS}$ and $c_{L,q}^{PS}$ with ref.[12]. Part of the discrepancies with ref.[10] have been explained above.

For the anomalous dimensions, we find at 1-loop,

$$\gamma_{qq}^{(0)}(N) = C_F \{-3 + 2S_1(N-1) + 2S_1(N+1)\}, \quad (121)$$

$$\gamma_{qg}^{(0)}(N) = n_f \{2S_1(N-1) + 8S_1(N+1) - 4S_1(N+2) - 6S_1(N)\}, \quad (122)$$

$$\gamma_{gq}^{(0)}(N) = C_F \{4S_1(N-2) - 8S_1(N-1) - 2S_1(N+1) + 6S_1(N)\}, \quad (123)$$

$$\gamma_{gg}^{(0)}(N) = C_A \{4S_1(N-2) - 8S_1(N-1) - 8S_1(N+1) + 4S_1(N+2) + 12S_1(N)\} - \beta_0. \quad (124)$$

At two loops, we obtain,

$$\begin{aligned} \gamma_{qq}^{(1),V}(N) = & (-1)^N \times \\ & \left[C_F C_A \left\{ (-1)^N \left(-\frac{17}{6} + 4\zeta_3 + \left(\frac{268}{9} - 8\zeta_2 \right) S_1(N-1) - \frac{44}{3} S_2(N-1) + 8S_3(N-1) \right) \right. \right. \\ & \left. \left. - 4S_{-3}(N-1) - 4S_{-3}(N+1) + 8S_{-3}(N) + \frac{10}{3} S_{-2}(N-1) + \frac{10}{3} S_{-2}(N+1) \right\} \right] \end{aligned} \quad (125)$$

$$\begin{aligned}
& -\frac{20}{3}S_{-2}(N) + \left(\frac{106}{9} + 4\zeta_2\right)S_{-1}(N-1) - \left(\frac{374}{9} - 4\zeta_2\right)S_{-1}(N+1) \\
& + \left(\frac{268}{9} - 8\zeta_2\right)S_{-1}(N) \Big\} \\
& + C_F n_f \left\{ (-1)^N \left(\frac{1}{3} - \frac{40}{9}S_1(N-1) + \frac{8}{3}S_2(N-1) \right) - \frac{4}{3}S_{-2}(N-1) - \frac{4}{3}S_{-2}(N+1) + \frac{8}{3}S_{-2}(N) \right. \\
& \quad \left. - \frac{4}{9}S_{-1}(N-1) + \frac{44}{9}S_{-1}(N+1) - \frac{40}{9}S_{-1}(N) \right\} \\
& + C_F^2 \left\{ (-1)^N \left(-\frac{3}{2} - 8\zeta_3 + 16S_1(N-1)\zeta_2 - 16S_{1,2}(N-1) + 12S_2(N-1) - 16S_{2,1}(N-1) \right) \right. \\
& \quad - 4S_{-3}(N-1) - 4S_{-3}(N+1) + 8S_{-3}(N) + 8S_{-2}(N+1) - 8S_{-2}(N) + 8S_{-2,1}(N-1) \\
& \quad + 8S_{-2,1}(N+1) - 16S_{-2,1}(N) - \left(20 + 8\zeta_2\right)S_{-1}(N-1) + \left(20 - 8\zeta_2\right)S_{-1}(N+1) \\
& \quad \left. + 16S_{-1}(N)\zeta_2 + 8S_{-1,2}(N-1) + 8S_{-1,2}(N+1) - 16S_{-1,2}(N) \right\} \Big],
\end{aligned}$$

$$\begin{aligned}
\gamma_{q\bar{q}}^{(1),V}(N) &= (-1)^N \times \\
& C_F \left(C_F - \frac{C_A}{2} \right) \left\{ 8\zeta_3 + 8S_{-3}(N-1) + 8S_{-3}(N+1) - 8S_{-2}(N-1) - 8S_{-2}(N+1) \right. \\
& \quad + 16S_{-2}(N) + 16S_{-1}(N-1) - 16S_{-1}(N+1) - 8S_1(N-1)\zeta_2 - 8S_1(N+1)\zeta_2 \\
& \quad \left. - 16S_{1,-2}(N-1) - 16S_{1,-2}(N+1) \right\},
\end{aligned} \tag{126}$$

$$\begin{aligned}
\gamma_{q\bar{q}}^{(1),S}(N) &= \gamma_{q\bar{q}}^{(1),S}(N) = (-1)^N \times \\
& C_F n_f \left\{ -4S_{-3}(N-1) - 4S_{-3}(N+1) + 8S_{-3}(N) - 2S_{-2}(N-1) - \frac{46}{3}S_{-2}(N+1) \right. \\
& \quad + \frac{16}{3}S_{-2}(N+2) + 12S_{-2}(N) - \frac{40}{9}S_{-1}(N-2) + \frac{4}{9}S_{-1}(N-1) - \frac{4}{9}S_{-1}(N+1) \\
& \quad \left. + \frac{112}{9}S_{-1}(N+2) - 8S_{-1}(N) \right\},
\end{aligned} \tag{127}$$

$$\begin{aligned}
\gamma_{qg}^{(1)}(N) &= (-1)^N \times \\
& \left[C_F n_f \left\{ 4S_{-3}(N-1) + 8S_{-3}(N+1) - 16S_{-3}(N+2) + 4S_{-3}(N) - 6S_{-2}(N-1) \right. \right. \\
& \quad - 8S_{-2}(N+1) + 16S_{-2}(N+2) - 2S_{-2}(N) - 8S_{-2,1}(N-1) + 16S_{-2,1}(N+2) \\
& \quad - 8S_{-2,1}(N) + 28S_{-1}(N-1) - 18S_{-1}(N+1) - 40S_{-1}(N+2) + 30S_{-1}(N) \\
& \quad - 16S_{-1,1}(N+2) + 16S_{-1,1}(N) + 8S_{-1,1,1}(N-1) - 16S_{-1,1,1}(N+2) + 8S_{-1,1,1}(N) \\
& \quad \left. \left. - 8S_{-1,2}(N-1) + 16S_{-1,2}(N+2) - 8S_{-1,2}(N) \right\} \right. \\
& + C_A n_f \left\{ -8S_{-3}(N-1) - 16S_{-3}(N+1) + 24S_{-3}(N) - 4S_{-2}(N-1) - \frac{272}{3}S_{-2}(N+1) \right. \\
& \quad \left. + \frac{176}{3}S_{-2}(N+2) + 36S_{-2}(N) - \frac{80}{9}S_{-1}(N-2) + \left(\frac{8}{9} + 4\zeta_2\right)S_{-1}(N-1) \right\}
\end{aligned} \tag{128}$$

$$\begin{aligned}
& + \frac{28}{9}S_{-1}(N+1) + \left(\frac{872}{9} - 8\zeta_2\right)S_{-1}(N+2) - \left(92 - 4\zeta_2\right)S_{-1}(N) + 16S_{-1,1}(N+2) \\
& - 16S_{-1,1}(N) - 8S_{-1,1,1}(N-1) + 16S_{-1,1,1}(N+2) - 8S_{-1,1,1}(N) - 4S_1(N-1)\zeta_2 \\
& - 16S_1(N+1)\zeta_2 + 8S_1(N+2)\zeta_2 + 12S_1(N)\zeta_2 - 8S_{1,-2}(N-1) - 32S_{1,-2}(N+1) \\
& + 16S_{1,-2}(N+2) + 24S_{1,-2}(N) \Bigg\} \Bigg],
\end{aligned}$$

$$\gamma_{\text{gq}}^{(1)}(N) = (-1)^N \times \quad (129)$$

$$\begin{aligned}
& \left[C_F C_A \left\{ 16S_{-3}(N-1) + 8S_{-3}(N+1) - 24S_{-3}(N) + 48S_{-2}(N-1) + \frac{92}{3}S_{-2}(N+1) \right. \right. \\
& - \frac{32}{3}S_{-2}(N+2) - 68S_{-2}(N) + 16S_{-2,1}(N-2) - 8S_{-2,1}(N+1) - 8S_{-2,1}(N) \\
& - \left(4 + 8\zeta_2\right)S_{-1}(N-2) + \frac{112}{9}S_{-1}(N-1) + \left(36 + 4\zeta_2\right)S_{-1}(N+1) - \frac{176}{9}S_{-1}(N+2) \\
& - \left(\frac{224}{9} - 4\zeta_2\right)S_{-1}(N) + \frac{88}{3}S_{-1,1}(N-2) - \frac{68}{3}S_{-1,1}(N+1) - \frac{20}{3}S_{-1,1}(N) \\
& - 16S_{-1,1,1}(N-2) + 8S_{-1,1,1}(N+1) + 8S_{-1,1,1}(N) + 16S_{-1,2}(N-2) - 8S_{-1,2}(N+1) \\
& - 8S_{-1,2}(N) - 8S_1(N-2)\zeta_2 + 16S_1(N-1)\zeta_2 + 4S_1(N+1)\zeta_2 - 12S_1(N)\zeta_2 \\
& \left. \left. - 16S_{1,-2}(N-2) + 32S_{1,-2}(N-1) + 8S_{1,-2}(N+1) - 24S_{1,-2}(N) \right\} \right. \\
& + C_F n_f \left\{ \frac{80}{9}S_{-1}(N-2) - \frac{64}{9}S_{-1}(N+1) - \frac{16}{9}S_{-1}(N) - \frac{16}{3}S_{-1,1}(N-2) + \frac{8}{3}S_{-1,1}(N+1) \right. \\
& \left. \left. + \frac{8}{3}S_{-1,1}(N) \right\} \right. \\
& + C_F^2 \left\{ -8S_{-3}(N-1) + 4S_{-3}(N+1) + 4S_{-3}(N) - 8S_{-2}(N-1) - 14S_{-2}(N+1) + 22S_{-2}(N) \right. \\
& - 10S_{-1}(N-1) - 14S_{-1}(N+1) + 24S_{-1}(N) - 24S_{-1,1}(N-2) + 20S_{-1,1}(N+1) \\
& \left. \left. + 4S_{-1,1}(N) + 16S_{-1,1,1}(N-2) - 8S_{-1,1,1}(N+1) - 8S_{-1,1,1}(N) \right\} \right],
\end{aligned}$$

$$\gamma_{\text{gg}}^{(1)}(N) = (-1)^N \times \quad (130)$$

$$\begin{aligned}
& \left[C_F n_f \left\{ 2(-1)^N - 8S_{-3}(N-1) - 8S_{-3}(N+1) + 16S_{-3}(N) + 12S_{-2}(N-1) + 20S_{-2}(N+1) \right. \right. \\
& - 32S_{-2}(N) - \frac{8}{3}S_{-1}(N-2) - \frac{88}{3}S_{-1}(N-1) + \frac{88}{3}S_{-1}(N+1) - \frac{40}{3}S_{-1}(N+2) \\
& \left. \left. + 16S_{-1}(N) \right\} \right. \\
& + C_A n_f \left\{ (-1)^N \left(\frac{8}{3} - \frac{40}{9}S_1(N-1) \right) + \frac{8}{3}S_{-2}(N-1) + \frac{8}{3}S_{-2}(N+1) - \frac{16}{3}S_{-2}(N) \right. \\
& \left. \left. + \frac{92}{9}S_{-1}(N-2) + \frac{8}{3}S_{-1}(N-1) + \frac{16}{9}S_{-1}(N+1) - \frac{92}{9}S_{-1}(N+2) - \frac{40}{9}S_{-1}(N) \right\} \right. \\
& + C_A^2 \left\{ (-1)^N \left(-\frac{32}{3} - 4\zeta_3 + \left(\frac{268}{9} + 8\zeta_2 \right) S_1(N-1) - 16S_{1,2}(N-1) - 16S_{2,1}(N-1) \right. \right. \\
& \left. \left. + 8S_3(N-1) \right) + 4\zeta_3 + 8S_{-3}(N-1) + 16S_{-3}(N+1) + 16S_{-3}(N+2) - 32S_{-3}(N) \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{100}{3} S_{-2}(N-1) + 44 S_{-2}(N+1) - \frac{176}{3} S_{-2}(N+2) - \frac{56}{3} S_{-2}(N) + 16 S_{-2,1}(N-2) \\
& + 16 S_{-2,1}(N-1) - 16 S_{-2,1}(N+2) - 16 S_{-2,1}(N) - 8 S_{-1}(N-2) \zeta_2 \\
& - \left(\frac{50}{9} + 8 \zeta_2 \right) S_{-1}(N-1) - \frac{218}{9} S_{-1}(N+1) + 8 S_{-1}(N+2) \zeta_2 + \left(\frac{268}{9} + 8 \zeta_2 \right) S_{-1}(N) \\
& + 16 S_{-1,2}(N-2) + 16 S_{-1,2}(N-1) - 16 S_{-1,2}(N+2) - 16 S_{-1,2}(N) - 8 S_1(N-2) \zeta_2 \\
& + 16 S_1(N-1) \zeta_2 + 16 S_1(N+1) \zeta_2 - 8 S_1(N+2) \zeta_2 - 24 S_1(N) \zeta_2 - 16 S_{1,-2}(N-2) \\
& + 32 S_{1,-2}(N-1) + 32 S_{1,-2}(N+1) - 16 S_{1,-2}(N+2) - 48 S_{1,-2}(N) \Bigg\} \Bigg] .
\end{aligned}$$

For the coefficient functions, we obtain at tree level,

$$c_{2,q}^{(0)}(N) = c_{3,q}^{(0)}(N) = 1, \quad c_{2,g}^{(0)}(N) = c_{L,q}^{(0)}(N) = c_{L,g}^{(0)}(N) = 0, \quad (131)$$

At 1-loop we find,

$$c_{2,q}^{(1)}(N) = C_F \{ -9 - 3 S_1(N-1) + 4 S_1(N+1) + 2 S_1(N) + 2 S_{1,1}(N-1) + 2 S_{1,1}(N+1) - 2 S_2(N-1) - 2 S_2(N+1) \}, \quad (132)$$

$$c_{3,q}^{(1)}(N) = c_{2,q}^{(1)}(N) + C_F \{ 2 S_1(N-1) - 2 S_1(N+1) \}, \quad (133)$$

$$\begin{aligned}
c_{2,g}^{(1)}(N) = n_f \{ & 2 S_1(N-1) + 32 S_1(N+1) - 16 S_1(N+2) - 18 S_1(N) + 2 S_{1,1}(N-1) \\
& + 8 S_{1,1}(N+1) - 4 S_{1,1}(N+2) - 6 S_{1,1}(N) - 2 S_2(N-1) - 8 S_2(N+1) \\
& + 4 S_2(N+2) + 6 S_2(N) \}, \quad (134)
\end{aligned}$$

$$c_{L,q}^{(1)}(N) = 4 C_F \{ S_1(N+1) - S_1(N) \}, \quad (135)$$

$$c_{L,g}^{(1)}(N) = 8 n_f \{ 2 S_1(N+1) - S_1(N+2) - S_1(N) \}. \quad (136)$$

Our results for the Mellin moments of 2-loop coefficient functions $c_{2,q}^{(2),\text{ns}}, c_{3,q}^{(2),\text{ns}}, c_{2,q}^{(2),\text{ps}}, c_{2,g}^{(2)}, c_{L,q}^{(2),\text{ns}}, c_{L,q}^{(2),\text{ps}}$ and $c_{L,g}^{(2)}$ can be found in appendix B.

4.3 Results in x -space

Let us now present our results in momentum space. The scale evolution of the DIS structure functions in x -space has been discussed in ref.[46]. It is governed by the Altarelli-Parisi splitting functions, defined as

$$\gamma_{ij}(N) = - \int_0^1 dx x^{N-1} P_{ij}(x), \quad (137)$$

where the conventional relation between the anomalous dimensions and the splitting functions in eq.(137) involves a relative sign. In the following, all expressions which diverge for $x \rightarrow 1$ are understood in the sense of $+$ -distributions as defined in eq.(90).

For the splitting functions, we find at 1-loop,

$$P_{qq}^{(0)}(x) = C_F \{ 2 p_{qq}(x) + 3 \delta(1-x) \}, \quad (138)$$

$$P_{qg}^{(0)}(x) = 2 n_f p_{qg}(x), \quad (139)$$

$$P_{gq}^{(0)}(x) = 2 C_F p_{gq}(x), \quad (140)$$

$$P_{gg}^{(0)}(x) = 4 C_A p_{gg}(x) + \beta_0 \delta(1-x), \quad (141)$$

where we have introduced the following polynomials,

$$p_{\text{qq}}(x) = \frac{2}{1-x} - 1 - x, \quad (142)$$

$$p_{\text{qg}}(x) = 1 - 2x + 2x^2, \quad (143)$$

$$p_{\text{gq}}(x) = \frac{2}{x} - 2 + x, \quad (144)$$

$$p_{\text{gg}}(x) = \frac{1}{1-x} + \frac{1}{x} - 2 + x - x^2. \quad (145)$$

At two loops, we obtain,²

$$P_{\text{qq}}^{(1),\text{V}}(x) = \quad (146)$$

$$\begin{aligned} & n_f C_F \left\{ -\frac{8}{3}(1-x) - \frac{4}{3} \left(\frac{5}{3} + H_0(x) \right) p_{\text{qq}}(x) - \left(\frac{1}{3} + \frac{8}{3} \zeta_2 \right) \delta(1-x) \right\} \\ & + C_F C_A \left\{ \frac{80}{3}(1-x) + 4(1+x)H_0(x) + 4 \left(\frac{67}{18} - \zeta_2 + \frac{11}{6} H_0(x) + H_{0,0}(x) \right) p_{\text{qq}}(x) \right. \\ & \quad \left. + \left(\frac{17}{6} + \frac{44}{3} \zeta_2 - 12 \zeta_3 \right) \delta(1-x) \right\} \\ & + C_F^2 \left\{ -20(1-x) - 2(3+7x)H_0(x) - 8 \left(\frac{3}{4} H_0(x) - H_{1,0}(x) - H_2(x) \right) p_{\text{qq}}(x) \right. \\ & \quad \left. - 4(1+x)H_{0,0}(x) + \left(\frac{3}{2} - 12 \zeta_2 + 24 \zeta_3 \right) \delta(1-x) \right\}, \end{aligned}$$

$$P_{\text{q}\bar{\text{q}}}^{(1),\text{V}}(x) = \quad (147)$$

$$C_F \left(C_F - \frac{C_A}{2} \right) \left\{ 16(1-x) + 8(1+x)H_0(x) + 8 \left(H_{0,0}(x) - 2H_{-1,0}(x) - \zeta_2 \right) p_{\text{qq}}(-x) \right\},$$

$$P_{\text{qq}}^{(1),\text{S}}(x) = P_{\text{q}\bar{\text{q}}}^{(1),\text{S}}(x) = \quad (148)$$

$$n_f C_F \left\{ -4 + \frac{40}{9x} + 12x - \frac{112}{9} x^2 - 4(1+x)H_{0,0}(x) + 2 \left(1 + 5x + \frac{8}{3} x^2 \right) H_0(x) \right\},$$

$$P_{\text{qg}}^{(1)}(x) = \quad (149)$$

$$\begin{aligned} & n_f C_F \left\{ 8 - 18x - 2(1-4x)H_0(x) - 8H_1(x) - 4(1-2x)H_{0,0}(x) \right. \\ & \quad \left. + 8 \left(\frac{5}{2} - \zeta_2 + H_0(x) + H_{0,0}(x) + H_1(x) + H_{1,0}(x) + H_{1,1}(x) + H_2(x) \right) p_{\text{qg}}(x) \right\} \\ & + n_f C_A \left\{ \frac{364}{9} + \frac{80}{9x} + \frac{28}{9} x - 8 \left(\frac{109}{18} - \frac{11}{3} H_0(x) + H_1(x) + H_{1,1}(x) \right) p_{\text{qg}}(x) \right. \\ & \quad \left. - \frac{4}{3} (19 - 68x)H_0(x) + 8H_1(x) - 8(1+2x)H_{0,0}(x) - 8H_{-1,0}(x)p_{\text{qg}}(-x) - 16x\zeta_2 \right\}, \end{aligned}$$

²The definition of S_2 in ref.[6] contains a typographical error. The lower integration boundary $(1+x)/x$ should read $x/(1+x)$, see for instance eq.(61) in ref.[8]. Also the timelike quark-quark splitting function in eq.(12) of ref.[6] contains a typographical mistake. The term $(10 - 18x - 16/3x^2) \ln(x)$ should read $(-10 - 18x - 16/3x^2) \ln(x)$.

$$\begin{aligned}
P_{\text{gq}}^{(1)}(x) = & \quad (150) \\
& n_f C_F \left\{ -\frac{8}{3}x - \frac{8}{3} \left(\frac{5}{3} - H_1(x) \right) p_{\text{gq}}(x) \right\} \\
& + C_F C_A \left\{ \frac{112}{9} + \frac{130}{9}x + \frac{176}{9}x^2 + 8 \left(\frac{1}{4} - \frac{11}{6}H_1(x) + H_{1,0}(x) + H_{1,1}(x) + H_2(x) \right) p_{\text{gq}}(x) \right. \\
& \quad \left. - 4 \left(12 + 5x + \frac{8}{3}x^2 \right) H_0(x) - 8xH_1(x) + 8(2+x)H_{0,0}(x) - 8H_{-1,0}(x)p_{\text{gq}}(-x) + 16\zeta_2 \right\} \\
& + C_F^2 \left\{ -10 - 14x - 4(2-x)H_{0,0}(x) + 2(4+7x)H_0(x) + 8xH_1(x) \right. \\
& \quad \left. + 8 \left(\frac{3}{2}H_1(x) - H_{1,1}(x) \right) p_{\text{gq}}(x) \right\},
\end{aligned}$$

$$\begin{aligned}
P_{\text{gg}}^{(1)}(x) = & \quad (151) \\
& n_f C_F \left\{ -32 + \frac{8}{3x} + 16x + \frac{40}{3}x^2 - 8(1+x)H_{0,0}(x) - 4(3+5x)H_0(x) - 2\delta(1-x) \right\} \\
& + n_f C_A \left\{ 4(1-x) - \frac{52}{9} \left(\frac{1}{x} - x^2 \right) - \frac{40}{9}p_{\text{gg}}(x) - \frac{8}{3}(1+x)H_0(x) - \frac{8}{3}\delta(1-x) \right\} \\
& + C_A^2 \left\{ 54(1-x) - \frac{268}{9} \left(\frac{1}{x} - x^2 \right) - \frac{4}{3}(25-11x+44x^2)H_0(x) + 32(1+x)H_{0,0}(x) \right. \\
& \quad + 8 \left(\frac{67}{18} - \zeta_2 + H_{0,0}(x) + 2H_{1,0}(x) + 2H_2(x) \right) p_{\text{gg}}(x) \\
& \quad \left. + 8 \left(H_{0,0}(x) - 2H_{-1,0}(x) - \zeta_2 \right) p_{\text{gg}}(-x) + \left(\frac{32}{3} + 12\zeta_3 \right) \delta(1-x) \right\}.
\end{aligned}$$

The x -space expressions for the coefficient functions are commonly defined by,

$$c_{i,p}(N) = \int_0^1 dx x^{N-1} c_{i,p}(x). \quad (152)$$

We obtain at tree level,

$$c_{2,q}^{(0)}(x) = c_{3,q}^{(0)}(x) = \delta(1-x), \quad c_{2,g}^{(0)}(x) = c_{L,q}^{(0)}(x) = c_{L,g}^{(0)}(x) = 0, \quad (153)$$

At 1-loop we find,

$$c_{2,q}^{(1)}(x) = C_F \left\{ \frac{9}{2} + \frac{5}{2}x - 2 \left(\frac{3}{4} + H_0(x) + H_1(x) \right) p_{\text{qq}}(x) - (9 + 4\zeta_2) \delta(1-x) \right\}, \quad (154)$$

$$c_{3,q}^{(1)}(x) = c_{2,q}^{(1)}(x) - 2C_F(1+x), \quad (155)$$

$$c_{2,g}^{(1)}(x) = n_f \left\{ 6 - 2 \left(4 + H_0(x) + H_1(x) \right) p_{\text{qg}}(x) \right\}, \quad (156)$$

$$c_{L,q}^{(1)}(x) = 4C_F x, \quad (157)$$

$$c_{L,g}^{(1)}(x) = 8n_f x(1-x). \quad (158)$$

The results for the 2-loop coefficient functions $c_{2,q}^{(2),\text{ns}}$, $c_{3,q}^{(2),\text{ns}}$, $c_{2,q}^{(2),\text{ps}}$, $c_{2,g}^{(2)}$, $c_{L,q}^{(2),\text{ns}}$, $c_{L,q}^{(2),\text{ps}}$ and $c_{L,g}^{(2)}$ have been deferred to appendix C.

5 Conclusions

In the present paper we have calculated the Mellin moments of the perturbative QCD corrections up to second order for the DIS structure functions F_2 , F_3 and F_L in leading twist approximation using the method of projection [32]. We have presented the analytic results for the 1- and 2-loop contributions to the anomalous dimensions of the singlet and non-singlet operator matrix elements and the 1- and 2-loop coefficient functions of F_2 , F_3 and F_L . Our results are in agreement with the literature as has been detailed in section 4.

Our choice to work in Mellin space enabled us to solve the 2-loop integrals in dimensional regularization, $D = 4 - 2\epsilon$, by means of recursion relations in the Mellin moment N . This approach has systematically extended previous work in this direction [10] and relied on the improved understanding of harmonic sums [4, 22, 23]. Progress was possible in particular due to the development of new algorithms for a large class of series that involve harmonic sums.

The method as we applied it turned out to be very flexible and in the expansion in ϵ it is in principle not limited to a certain order. This allowed us to calculate anomalous dimensions and coefficient functions at the same time. As a byproduct, we have also performed the calculation of all $\mathcal{O}(\epsilon^2)$ -terms at 1-loops and all $\mathcal{O}(\epsilon)$ -terms at two loops, for all unpolarized structure functions, as needed to extract the coefficient functions at 3-loops after mass factorization. However these results will be published elsewhere [47].

We have been able to perform the inverse Mellin transformation of our N -space results analytically and we have given the corresponding expressions in x -space as well. Our x -space results are expressed in terms of harmonic polylogarithms, which we believe to be the natural class of functions for calculations of DIS structure functions. Again, this has been achieved due to deeper insight gained into the subtle interplay between harmonic sums and harmonic polylogarithms [24].

We are very confident, that the program presented in this work can be successfully applied to the ultimate goal, the calculation of the anomalous dimensions and the coefficient functions of the DIS structure functions at 3-loops. Finally we would like to note that a calculation of the perturbative QCD corrections to the polarized structure function g_1 up to second order and in leading twist approximation is in progress [47].

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Appendix A

Here we give the results for the harmonic polylogarithms of weight two and three in terms of standard polylogarithms,

$$\text{Li}_2(x) = - \int_0^x \frac{dz}{z} \ln(1-z), \quad \text{Li}_3(x) = \int_0^x \frac{dz}{z} \text{Li}_2(z), \quad (159)$$

where $\text{Li}_2(x)$ is Euler's dilogarithm and $\text{Li}_3(x)$ the usual trilogarithm, see ref.[48]. For It is obvious, that we only need to consider a limited subset at weight two and three. We find at weight two,

$$H_{-1,-1}(x) = \frac{1}{2} \ln^2(1+x), \quad (160)$$

$$H_{-1,0}(x) = \ln x \ln(1+x) + \text{Li}_2(-x), \quad (161)$$

$$H_{-1,1}(x) = -\frac{1}{2}\zeta_2 + \frac{1}{2}\ln^2 2 - \ln(1+x)\ln 2 + \text{Li}_2\left(\frac{1+x}{2}\right), \quad (162)$$

$$H_{0,-1}(x) = -\text{Li}_2(-x);, \quad (163)$$

$$H_{0,0}(x) = \frac{1}{2}\ln^2 x, \quad (164)$$

$$H_{0,1}(x) = \text{Li}_2(x), \quad (165)$$

$$H_{1,-1}(x) = \frac{1}{2}\zeta_2 - \frac{1}{2}\ln^2 2 - \ln(1-x)\ln(1+x) + \ln(1+x)\ln 2 - \text{Li}_2\left(\frac{1+x}{2}\right), \quad (166)$$

$$H_{1,0}(x) = -\ln x \ln(1-x) - \text{Li}_2(x), \quad (167)$$

$$H_{1,1}(x) = \frac{1}{2}\ln^2(1-x). \quad (168)$$

At weight three we obtain,

$$H_{-1,-1,-1}(x) = \frac{1}{6}\ln^3(1+x), \quad (169)$$

$$H_{-1,-1,0}(x) = \zeta_3 - \ln(1+x)\zeta_2 + \frac{1}{6}\ln^3(1+x) - \text{Li}_3\left(\frac{1}{1+x}\right), \quad (170)$$

$$H_{-1,-1,1}(x) = \frac{1}{2}\zeta_2 \ln 2 - \frac{7}{8}\zeta_3 - \frac{1}{6}\ln^3 2 - \frac{1}{2}\ln(1+x)\zeta_2 + \frac{1}{2}\ln(1+x)\ln^2 2 - \frac{1}{2}\ln^2(1+x)\ln 2 + \text{Li}_3\left(\frac{1+x}{2}\right), \quad (171)$$

$$H_{-1,0,-1}(x) = -2\zeta_3 + 2\ln(1+x)\zeta_2 - \frac{1}{3}\ln^3(1+x) + \ln(1+x)\text{Li}_2(-x) + \ln x \ln^2(1+x) + 2\text{Li}_3\left(\frac{1}{1+x}\right), \quad (172)$$

$$H_{-1,0,0}(x) = \frac{1}{2}\ln^2 x \ln(1+x) + \ln x \text{Li}_2(-x) - \text{Li}_3(-x), \quad (173)$$

$$H_{-1,0,1}(x) = \frac{1}{4}\zeta_3 - \frac{1}{2}\ln(1-x)\zeta_2 - \ln(1-x)\text{Li}_2(-x) + \frac{1}{2}\ln(1+x)\zeta_2 + \frac{1}{6}\ln^3(1+x) - \ln x \ln(1-x)\ln(1+x) - \text{Li}_3(1-x) - \text{Li}_3\left(-\frac{1-x}{1+x}\right) - \text{Li}_3\left(\frac{1}{1+x}\right) + \text{Li}_3\left(\frac{1-x}{1+x}\right), \quad (174)$$

$$H_{-1,1,-1}(x) = -\zeta_2 \ln 2 + \frac{7}{4}\zeta_3 + \frac{1}{3}\ln^3 2 + \frac{1}{2}\ln(1+x)\zeta_2 - \frac{1}{2}\ln(1+x)\ln^2 2 + \ln(1+x)\text{Li}_2\left(\frac{1+x}{2}\right) - 2\text{Li}_3\left(\frac{1+x}{2}\right), \quad (175)$$

$$H_{-1,1,0}(x) = -\frac{1}{2}\zeta_2 \ln 2 - \frac{1}{8}\zeta_3 + \frac{1}{6}\ln^3 2 + \frac{3}{2}\ln(1+x)\zeta_2 - \frac{1}{2}\ln(1+x)\ln^2 2 + \frac{1}{2}\ln^2(1+x)\ln 2 - \frac{1}{3}\ln^3(1+x) - \frac{1}{2}\ln x \zeta_2 + \frac{1}{2}\ln x \ln^2 2 - \ln x \ln(1+x)\ln 2 + \frac{1}{2}\ln x \ln^2(1+x) + \ln x \text{Li}_2\left(\frac{1+x}{2}\right) - \text{Li}_3\left(\frac{1+x}{2}\right) + \text{Li}_3(-x) + \text{Li}_3\left(\frac{1}{1+x}\right) + \text{Li}_3\left(\frac{2x}{1+x}\right) - \text{Li}_3(x), \quad (176)$$

$$H_{-1,1,1}(x) = \frac{1}{2}\zeta_2 \ln 2 - \frac{1}{8}\zeta_3 - \frac{1}{6}\ln^3 2 + \ln(1-x)\zeta_2 - \frac{1}{2}\ln(1-x)\ln^2 2 + \frac{1}{2}\ln^2(1-x)\ln 2 - \frac{1}{2}\ln^2(1-x)\ln(1+x) + \frac{1}{2}\ln(1-x)\ln^2(1+x) - \ln(1-x)\text{Li}_2\left(\frac{1+x}{2}\right) - \ln(1+x)\zeta_2 + \frac{1}{2}\ln(1+x)\ln^2 2 - \frac{1}{6}\ln^3(1+x) \quad (177)$$

$$\begin{aligned}
& +\text{Li}_3\left(\frac{1+x}{2}\right) + \text{Li}_3\left(-\frac{1-x}{1+x}\right), \\
H_{0,-1,-1}(x) &= \zeta_3 - \ln(1+x)\zeta_2 + \frac{1}{6}\ln^3(1+x) - \ln(1+x)\text{Li}_2(-x) - \frac{1}{2}\ln x \ln^2(1+x) \\
& - \text{Li}_3\left(\frac{1}{1+x}\right) \\
&= \text{S}_{n,p}(-x),
\end{aligned} \tag{178}$$

$$H_{0,-1,0}(x) = -\ln x \text{Li}_2(-x) + 2\text{Li}_3(-x), \tag{179}$$

$$H_{0,-1,1}(x) = \frac{1}{2}\zeta_2 \ln 2 - \frac{1}{8}\zeta_3 - \frac{1}{6}\ln^3 2 + \frac{1}{2}\ln(1-x)\zeta_2 + \ln(1-x)\text{Li}_2(-x) \tag{180}$$

$$\begin{aligned}
& -2\ln(1+x)\zeta_2 + \frac{1}{2}\ln(1+x)\ln^2 2 - \frac{1}{2}\ln^2(1+x)\ln 2 + \frac{1}{6}\ln^3(1+x) \\
& + \ln x \ln(1-x)\ln(1+x) - \frac{1}{2}\ln x \ln^2(1+x) + \text{Li}_3\left(\frac{1+x}{2}\right) + \text{Li}_3(1-x) \\
& - \text{Li}_3(-x) + \text{Li}_3\left(-\frac{1-x}{1+x}\right) - \text{Li}_3\left(\frac{1-x}{1+x}\right) - \text{Li}_3\left(\frac{2x}{1+x}\right) + \text{Li}_3(x), \\
H_{0,0,-1}(x) &= -\text{Li}_3(-x),
\end{aligned} \tag{181}$$

$$H_{0,0,0}(x) = \frac{1}{6}\ln^3 x, \tag{182}$$

$$H_{0,0,1}(x) = \text{Li}_3(x), \tag{183}$$

$$\begin{aligned}
H_{0,1,-1}(x) &= -\frac{1}{2}\zeta_2 \ln 2 - \frac{1}{8}\zeta_3 + \frac{1}{6}\ln^3 2 + \frac{3}{2}\ln(1+x)\zeta_2 - \frac{1}{2}\ln(1+x)\ln^2 2 \\
& + \frac{1}{2}\ln^2(1+x)\ln 2 - \frac{1}{3}\ln^3(1+x) + \ln(1+x)\text{Li}_2(x) + \frac{1}{2}\ln x \ln^2(1+x) \\
& - \text{Li}_3\left(\frac{1+x}{2}\right) + \text{Li}_3(-x) + \text{Li}_3\left(\frac{1}{1+x}\right) + \text{Li}_3\left(\frac{2x}{1+x}\right) - \text{Li}_3(x), \\
H_{0,1,0}(x) &= \ln x \text{Li}_2(x) - 2\text{Li}_3(x),
\end{aligned} \tag{184}$$

$$H_{0,1,1}(x) = \zeta_3 + \ln(1-x)\zeta_2 - \ln(1-x)\text{Li}_2(x) - \frac{1}{2}\ln x \ln^2(1-x) - \text{Li}_3(1-x), \tag{185}$$

$$\begin{aligned}
H_{1,-1,-1}(x) &= \frac{1}{2}\zeta_2 \ln 2 - \frac{7}{8}\zeta_3 - \frac{1}{6}\ln^3 2 - \frac{1}{2}\ln(1-x)\ln^2(1+x) + \frac{1}{2}\ln^2(1+x)\ln 2 \\
& - \ln(1+x)\text{Li}_2\left(\frac{1+x}{2}\right) + \text{Li}_3\left(\frac{1+x}{2}\right), \\
H_{1,-1,0}(x) &= \frac{1}{2}\zeta_2 \ln 2 - \frac{1}{8}\zeta_3 - \frac{1}{6}\ln^3 2 + \frac{1}{2}\ln(1-x)\zeta_2 - 2\ln(1+x)\zeta_2
\end{aligned} \tag{186}$$

$$\begin{aligned}
& + \frac{1}{2}\ln(1+x)\ln^2 2 - \frac{1}{2}\ln^2(1+x)\ln 2 + \frac{1}{6}\ln^3(1+x) + \frac{1}{2}\ln x \zeta_2 - \frac{1}{2}\ln x \ln^2 2 \\
& + \ln x \ln(1+x)\ln 2 - \frac{1}{2}\ln x \ln^2(1+x) - \ln x \text{Li}_2\left(\frac{1+x}{2}\right) + \text{Li}_3\left(\frac{1+x}{2}\right) \\
& + \text{Li}_3(1-x) - \text{Li}_3(-x) + \text{Li}_3\left(-\frac{1-x}{1+x}\right) - \text{Li}_3\left(\frac{1-x}{1+x}\right) - \text{Li}_3\left(\frac{2x}{1+x}\right) + \text{Li}_3(x), \\
H_{1,-1,1}(x) &= -\zeta_2 \ln 2 + \frac{1}{4}\zeta_3 + \frac{1}{3}\ln^3 2 - \frac{3}{2}\ln(1-x)\zeta_2 + \frac{1}{2}\ln(1-x)\ln^2 2
\end{aligned} \tag{187}$$

$$\begin{aligned}
& - \ln^2(1-x)\ln 2 + \ln^2(1-x)\ln(1+x) + \ln(1-x)\ln(1+x)\ln 2 \\
& - \ln(1-x)\ln^2(1+x) + \ln(1-x)\text{Li}_2\left(\frac{1+x}{2}\right) + 2\ln(1+x)\zeta_2 - \ln(1+x)\ln^2 2 \\
& + \frac{1}{3}\ln^3(1+x) - 2\text{Li}_3\left(\frac{1+x}{2}\right) - 2\text{Li}_3\left(-\frac{1-x}{1+x}\right), \\
H_{1,0,-1}(x) &= \frac{1}{4}\zeta_3 - \frac{1}{2}\ln(1-x)\zeta_2 + \frac{1}{2}\ln(1+x)\zeta_2 + \frac{1}{6}\ln^3(1+x) - \ln(1+x)\text{Li}_2(x)
\end{aligned} \tag{188}$$

$$\begin{aligned}
& - \ln^2(1-x)\ln 2 + \ln^2(1-x)\ln(1+x) + \ln(1-x)\ln(1+x)\ln 2 \\
& - \ln(1-x)\ln^2(1+x) + \ln(1-x)\text{Li}_2\left(\frac{1+x}{2}\right) + 2\ln(1+x)\zeta_2 - \ln(1+x)\ln^2 2 \\
& + \frac{1}{3}\ln^3(1+x) - 2\text{Li}_3\left(\frac{1+x}{2}\right) - 2\text{Li}_3\left(-\frac{1-x}{1+x}\right), \\
H_{1,0,1}(x) &= \frac{1}{4}\zeta_3 - \frac{1}{2}\ln(1-x)\zeta_2 + \frac{1}{2}\ln(1+x)\zeta_2 + \frac{1}{6}\ln^3(1+x) - \ln(1+x)\text{Li}_2(x)
\end{aligned} \tag{189}$$

$$H_{1,0,-1}(x) = \frac{1}{4}\zeta_3 - \frac{1}{2}\ln(1-x)\zeta_2 + \frac{1}{2}\ln(1+x)\zeta_2 + \frac{1}{6}\ln^3(1+x) - \ln(1+x)\text{Li}_2(x) \tag{190}$$

$$\begin{aligned}
& -\ln x \ln(1-x) \ln(1+x) - \text{Li}_3(1-x) - \text{Li}_3\left(-\frac{1-x}{1+x}\right) - \text{Li}_3\left(\frac{1}{1+x}\right) \\
& + \text{Li}_3\left(\frac{1-x}{1+x}\right),
\end{aligned}$$

$$\begin{aligned}
H_{1,0,0}(x) &= -\frac{1}{2} \ln^2 x \ln(1-x) - \ln x \text{Li}_2(x) + \text{Li}_3(x) \\
&= \zeta_3 - S_{n,p}(1-x),
\end{aligned} \tag{191}$$

$$H_{1,0,1}(x) = -2\zeta_3 - 2\ln(1-x)\zeta_2 + \ln(1-x)\text{Li}_2(x) + \ln x \ln^2(1-x) + 2\text{Li}_3(1-x), \tag{192}$$

$$\begin{aligned}
H_{1,1,-1}(x) &= \frac{1}{2}\zeta_2 \ln 2 - \frac{1}{8}\zeta_3 - \frac{1}{6}\ln^3 2 + \frac{1}{2}\ln(1-x)\zeta_2 + \frac{1}{2}\ln^2(1-x) \ln 2 \\
& - \ln(1-x) \ln(1+x) \ln 2 + \frac{1}{2}\ln(1-x) \ln^2(1+x) - \ln(1+x)\zeta_2 + \frac{1}{2}\ln(1+x) \ln^2 2 \\
& - \frac{1}{6}\ln^3(1+x) + \text{Li}_3\left(\frac{1+x}{2}\right) + \text{Li}_3\left(-\frac{1-x}{1+x}\right),
\end{aligned} \tag{193}$$

$$H_{1,1,0}(x) = \zeta_3 + \ln(1-x)\zeta_2 - \text{Li}_3(1-x), \tag{194}$$

$$H_{1,1,1}(x) = -\frac{1}{6}\ln^3(1-x). \tag{195}$$

The function $S_{n,p}$ in eqs.(178) and (191) denote the Nielsen functions [48], defined as

$$S_{n,p}(x) = \frac{(-1)^{p+n-1}}{p!(n-1)!} \int_0^1 \frac{dz}{z} \ln^{n-1}(z) \ln^p(1-xz). \tag{196}$$

Appendix B

Here we present the formulae for the Mellin moments of the 2-loop coefficient functions. We obtain,

$$\begin{aligned}
c_{2,q}^{(2),+\text{ns}}(N) &= \delta(N-2) \left\{ \left(\frac{11581}{360} - \frac{86}{9}\zeta_2 - \frac{32}{5}\zeta_2^2 - \frac{9}{5}\zeta_3 \right) C_F C_A - 4C_F n_f \right. \\
& \quad \left. - \left(\frac{24359}{1620} - \frac{172}{9}\zeta_2 - \frac{64}{5}\zeta_2^2 + \frac{142}{5}\zeta_3 \right) C_F^2 \right\} + \theta(N-3) (-1)^N \times \\
& \quad \left[C_F C_A \left\{ (-1)^N \left(-\frac{5465}{72} - 4\zeta_2 - \frac{32}{5}\zeta_2^2 + 54\zeta_3 - 12S_{-2}(N-1)\zeta_2 - 24S_{-2,-2}(N-1) \right. \right. \right. \\
& \quad + \left(\frac{3155}{54} - 32\zeta_3 \right) S_1(N-1) + \left(\frac{367}{9} - 16\zeta_2 \right) S_{1,1}(N-1) + \frac{44}{3} S_{1,1,1}(N-1) \\
& \quad + 8S_{1,1,2}(N-1) - \frac{44}{3} S_{1,2}(N-1) - 8S_{1,2,1}(N-1) + 16S_{1,3}(N-1) \\
& \quad - \left(\frac{239}{3} - 12\zeta_2 \right) S_2(N-1) - \frac{88}{3} S_{2,1}(N-1) + \frac{110}{3} S_3(N-1) + 8S_{3,1}(N-1) \\
& \quad - 12S_4(N-1) \Big) + 6S_{-4}(N-1) + 6S_{-4}(N+1) - 12S_{-4}(N) - \frac{43}{3} S_{-3}(N-1) \\
& \quad + \frac{5}{3} S_{-3}(N+1) - \frac{156}{5} S_{-3}(N+2) + \frac{36}{5} S_{-3}(N+3) + \frac{110}{3} S_{-3}(N) - 4S_{-3,1}(N-1) \\
& \quad - 4S_{-3,1}(N+1) + 8S_{-3,1}(N) - \frac{4}{5} S_{-2}(N-2) + \left(\frac{616}{15} - 4\zeta_2 \right) S_{-2}(N-1) \\
& \quad + \left(\frac{1366}{15} - 16\zeta_2 \right) S_{-2}(N+1) + \frac{36}{5} S_{-2}(N+2) - \left(\frac{2078}{15} - 20\zeta_2 \right) S_{-2}(N) \\
& \quad + \frac{32}{3} S_{-2,1}(N-1) + \frac{32}{3} S_{-2,1}(N+1) - \frac{64}{3} S_{-2,1}(N) - \frac{2}{5} S_{-1}(N-3)\zeta_2 \\
& \quad \left. \left. - \left(\frac{4}{5} - \frac{2}{5}\zeta_2 \right) S_{-1}(N-2) - \left(\frac{1414}{135} + 8\zeta_2 - 6\zeta_3 \right) S_{-1}(N-1) \right\} \right]
\end{aligned} \tag{197}$$

$$\begin{aligned}
& -\left(\frac{17761}{135} + 4\zeta_2 - 66\zeta_3\right)S_{-1}(N+1) - \left(\frac{36}{5} - \frac{78}{5}\zeta_2\right)S_{-1}(N+2) - \frac{18}{5}S_{-1}(N+3)\zeta_2 \\
& + \left(\frac{4051}{27} - 72\zeta_3\right)S_{-1}(N) + \left(\frac{16}{9} + 4\zeta_2\right)S_{-1,1}(N-1) - \left(\frac{740}{9} - 28\zeta_2\right)S_{-1,1}(N+1) \\
& + \left(\frac{724}{9} - 32\zeta_2\right)S_{-1,1}(N) - \frac{22}{3}S_{-1,1,1}(N-1) - \frac{22}{3}S_{-1,1,1}(N+1) + \frac{44}{3}S_{-1,1,1}(N) \\
& - 4S_{-1,1,2}(N-1) - 4S_{-1,1,2}(N+1) + 8S_{-1,1,2}(N) + \frac{22}{3}S_{-1,2}(N-1) + \frac{22}{3}S_{-1,2}(N+1) \\
& - \frac{44}{3}S_{-1,2}(N) + 4S_{-1,2,1}(N-1) + 4S_{-1,2,1}(N+1) - 8S_{-1,2,1}(N) - 4S_{-1,3}(N-1) \\
& - 28S_{-1,3}(N+1) + 32S_{-1,3}(N) + \frac{2}{5}S_1(N-3)\zeta_2 - \frac{2}{5}S_1(N-2)\zeta_2 + 10S_1(N-1)\zeta_2 \\
& - 10S_1(N+1)\zeta_2 + \frac{78}{5}S_1(N+2)\zeta_2 - \frac{18}{5}S_1(N+3)\zeta_2 - 12S_1(N)\zeta_2 + \frac{4}{5}S_{1,-2}(N-3) \\
& - \frac{4}{5}S_{1,-2}(N-2) + 20S_{1,-2}(N-1) - 20S_{1,-2}(N+1) + \frac{156}{5}S_{1,-2}(N+2) \\
& - \frac{36}{5}S_{1,-2}(N+3) - 24S_{1,-2}(N) + 4S_2(N-1)\zeta_2 + 16S_2(N+1)\zeta_2 - 20S_2(N)\zeta_2 \\
& + 8S_{2,-2}(N-1) + 32S_{2,-2}(N+1) - 40S_{2,-2}(N) \Big\} \\
& + C_{Fnf} \Big\{ (-1)^N \left(\frac{457}{36} - \frac{247}{27}S_1(N-1) - \frac{58}{9}S_{1,1}(N-1) - \frac{8}{3}S_{1,1,1}(N-1) + \frac{8}{3}S_{1,2}(N-1) \right. \\
& + \frac{38}{3}S_2(N-1) + \frac{16}{3}S_{2,1}(N-1) - \frac{20}{3}S_3(N-1) \Big) + \frac{10}{3}S_{-3}(N-1) + \frac{10}{3}S_{-3}(N+1) \\
& - \frac{20}{3}S_{-3}(N) - \frac{26}{3}S_{-2}(N-1) - \frac{38}{3}S_{-2}(N+1) + \frac{64}{3}S_{-2}(N) - \frac{8}{3}S_{-2,1}(N-1) \\
& - \frac{8}{3}S_{-2,1}(N+1) + \frac{16}{3}S_{-2,1}(N) + \frac{158}{27}S_{-1}(N-1) + \frac{488}{27}S_{-1}(N+1) - \frac{646}{27}S_{-1}(N) \\
& + \frac{32}{9}S_{-1,1}(N-1) + \frac{68}{9}S_{-1,1}(N+1) - \frac{100}{9}S_{-1,1}(N) + \frac{4}{3}S_{-1,1,1}(N-1) \\
& \left. + \frac{4}{3}S_{-1,1,1}(N+1) - \frac{8}{3}S_{-1,1,1}(N) - \frac{4}{3}S_{-1,2}(N-1) - \frac{4}{3}S_{-1,2}(N+1) + \frac{8}{3}S_{-1,2}(N) \right\} \\
& + C_F^2 \Big\{ (-1)^N \left(\frac{331}{8} + 8\zeta_2 + \frac{64}{5}\zeta_2^2 - 72\zeta_3 + 24S_{-2}(N-1)\zeta_2 + 48S_{-2,-2}(N-1) \right. \\
& - \left(\frac{51}{2} - 16\zeta_3 \right) S_1(N-1) - \left(27 - 32\zeta_2 \right) S_{1,1}(N-1) + 36S_{1,1,1}(N-1) \\
& + 48S_{1,1,1,1}(N-1) - 64S_{1,1,2}(N-1) - 36S_{1,2}(N-1) - 48S_{1,2,1}(N-1) + 24S_{1,3}(N-1) \\
& + \left(61 - 24\zeta_2 \right) S_2(N-1) - 24S_{2,1}(N-1) - 56S_{2,1,1}(N-1) + 48S_{2,2}(N-1) \\
& + 6S_3(N-1) + 48S_{3,1}(N-1) - 16S_4(N-1) \Big) + 18S_{-4}(N-1) + 18S_{-4}(N+1) \\
& - 36S_{-4}(N) - 32S_{-3}(N-1) - 76S_{-3}(N+1) + \frac{312}{5}S_{-3}(N+2) - \frac{72}{5}S_{-3}(N+3) \\
& + 60S_{-3}(N) - 32S_{-3,1}(N-1) - 32S_{-3,1}(N+1) + 64S_{-3,1}(N) + \frac{8}{5}S_{-2}(N-2) \\
& - \left(\frac{154}{5} - 8\zeta_2 \right) S_{-2}(N-1) - \left(\frac{284}{5} - 32\zeta_2 \right) S_{-2}(N+1) - \frac{72}{5}S_{-2}(N+2) \\
& + \left(\frac{502}{5} - 40\zeta_2 \right) S_{-2}(N) + 40S_{-2,1}(N-1) + 56S_{-2,1}(N+1) - 96S_{-2,1}(N) \\
& \left. + 32S_{-2,1,1}(N-1) + 32S_{-2,1,1}(N+1) - 64S_{-2,1,1}(N) - 28S_{-2,2}(N-1) \right\}
\end{aligned}$$

$$\begin{aligned}
& -28S_{-2,2}(N+1) + 56S_{-2,2}(N) + \frac{4}{5}S_{-1}(N-3)\zeta_2 + \left(\frac{8}{5} - \frac{4}{5}\zeta_2\right)S_{-1}(N-2) \\
& -\left(\frac{46}{5} - 16\zeta_2 - 12\zeta_3\right)S_{-1}(N-1) + \left(\frac{471}{5} + 8\zeta_2 - 108\zeta_3\right)S_{-1}(N+1) \\
& +\left(\frac{72}{5} - \frac{156}{5}\zeta_2\right)S_{-1}(N+2) + \frac{36}{5}S_{-1}(N+3)\zeta_2 - \left(101 - 96\zeta_3\right)S_{-1}(N) \\
& -\left(32 + 8\zeta_2\right)S_{-1,1}(N-1) + \left(84 - 56\zeta_2\right)S_{-1,1}(N+1) - \left(52 - 64\zeta_2\right)S_{-1,1}(N) \\
& -28S_{-1,1,1}(N-1) - 36S_{-1,1,1}(N+1) + 64S_{-1,1,1}(N) - 24S_{-1,1,1,1}(N-1) \\
& -24S_{-1,1,1,1}(N+1) + 48S_{-1,1,1,1}(N) + 32S_{-1,1,2}(N-1) + 32S_{-1,1,2}(N+1) \\
& -64S_{-1,1,2}(N) + 32S_{-1,2}(N-1) + 32S_{-1,2}(N+1) - 64S_{-1,2}(N) + 24S_{-1,2,1}(N-1) \\
& +24S_{-1,2,1}(N+1) - 48S_{-1,2,1}(N) - 20S_{-1,3}(N-1) + 28S_{-1,3}(N+1) - 8S_{-1,3}(N) \\
& -\frac{4}{5}S_1(N-3)\zeta_2 + \frac{4}{5}S_1(N-2)\zeta_2 - 20S_1(N-1)\zeta_2 + 20S_1(N+1)\zeta_2 - \frac{156}{5}S_1(N+2)\zeta_2 \\
& +\frac{36}{5}S_1(N+3)\zeta_2 + 24S_1(N)\zeta_2 - \frac{8}{5}S_{1,-2}(N-3) + \frac{8}{5}S_{1,-2}(N-2) - 40S_{1,-2}(N-1) \\
& +40S_{1,-2}(N+1) - \frac{312}{5}S_{1,-2}(N+2) + \frac{72}{5}S_{1,-2}(N+3) + 48S_{1,-2}(N) - 8S_2(N-1)\zeta_2 \\
& -32S_2(N+1)\zeta_2 + 40S_2(N)\zeta_2 - 16S_{2,-2}(N-1) - 64S_{2,-2}(N+1) + 80S_{2,-2}(N) \Bigg] ,
\end{aligned}$$

$$c_{2,q}^{(2),-\text{ns}}(N) = \quad (198)$$

$$\begin{aligned}
& \delta(N-2) \left\{ \left(\frac{5327}{540} - \frac{172}{9}\zeta_2 - \frac{64}{5}\zeta_2^2 + \frac{238}{5}\zeta_3 \right) C_F \left(C_F - \frac{C_A}{2} \right) \right\} + \theta(N-3) (-1)^N \times \\
& \left[C_F \left(C_F - \frac{C_A}{2} \right) \left\{ -8\zeta_2 - \frac{64}{5}\zeta_2^2 - 12S_{-4}(N-1) - 12S_{-4}(N+1) + 8S_{-3}(N-1) \right. \right. \\
& +80S_{-3}(N+1) - \frac{168}{5}S_{-3}(N+2) - \frac{72}{5}S_{-3}(N+3) - 40S_{-3}(N) + 8S_{-3,1}(N-1) \\
& +8S_{-3,1}(N+1) + \frac{8}{5}S_{-2}(N-2) - \left(\frac{74}{5} + 16\zeta_2 \right) S_{-2}(N-1) - \left(\frac{74}{5} + 32\zeta_2 \right) S_{-2}(N+1) \\
& -\frac{72}{5}S_{-2}(N+2) + \left(\frac{132}{5} + 24\zeta_2 \right) S_{-2}(N) - 8S_{-2,1}(N-1) - 8S_{-2,1}(N+1) \\
& +16S_{-2,1}(N) + \frac{4}{5}S_{-1}(N-3)\zeta_2 + \left(\frac{8}{5} - \frac{4}{5}\zeta_2 \right) S_{-1}(N-2) + \left(\frac{154}{5} + 20\zeta_2 \right) S_{-1}(N-1) \\
& -\left(\frac{154}{5} + 28\zeta_2 \right) S_{-1}(N+1) + \left(\frac{72}{5} + \frac{84}{5}\zeta_2 \right) S_{-1}(N+2) + \frac{36}{5}S_{-1}(N+3)\zeta_2 \\
& -\left(16 + 16\zeta_2 \right) S_{-1}(N) + 16S_{-1,1}(N-1) - 16S_{-1,1}(N+1) - \frac{4}{5}S_1(N-3)\zeta_2 \\
& +\frac{4}{5}S_1(N-2)\zeta_2 - \left(16\zeta_2 - 20\zeta_3 \right) S_1(N-1) - \left(40\zeta_2 - 36\zeta_3 \right) S_1(N+1) + \frac{84}{5}S_1(N+2)\zeta_2 \\
& +\frac{36}{5}S_1(N+3)\zeta_2 + \left(32\zeta_2 - 24\zeta_3 \right) S_1(N) + 48S_{1,-3}(N-1) + 80S_{1,-3}(N+1) \\
& -48S_{1,-3}(N) - \frac{8}{5}S_{1,-2}(N-3) + \frac{8}{5}S_{1,-2}(N-2) - 32S_{1,-2}(N-1) - 80S_{1,-2}(N+1) \\
& +\frac{168}{5}S_{1,-2}(N+2) + \frac{72}{5}S_{1,-2}(N+3) + 64S_{1,-2}(N) - 16S_{1,-2,1}(N-1) \\
& -16S_{1,-2,1}(N+1) - 24S_{1,1}(N-1)\zeta_2 - 56S_{1,1}(N+1)\zeta_2 + 48S_{1,1}(N)\zeta_2 \\
& -48S_{1,1,-2}(N-1) - 112S_{1,1,-2}(N+1) + 96S_{1,1,-2}(N) + 16S_2(N-1)\zeta_2
\end{aligned}$$

$$+32S_2(N+1)\zeta_2 - 24S_2(N)\zeta_2 + 32S_{2,-2}(N-1) + 64S_{2,-2}(N+1) - 48S_{2,-2}(N)\Bigg\},$$

$$\begin{aligned}
c_{3,q}^{(2),+\text{ns}}(N) = & c_{2,q}^{(2),+\text{ns}}(N) + \delta(N-1)\left\{\left(-\frac{175}{8} - 4\zeta_2 - \frac{32}{5}\zeta_2^2 + 19\zeta_3\right)C_FC_A + 4C_Fn_f \right. \\
& + \left(\frac{33}{4} + 8\zeta_2 + \frac{64}{5}\zeta_2^2 - 38\zeta_3\right)C_F^2\Bigg\} + \delta(N-2)\left\{\left(-\frac{13669}{540} + \frac{1}{3}\zeta_2 + \frac{44}{5}\zeta_3\right)C_FC_A \right. \\
& + \frac{106}{27}C_Fn_f + \left(\frac{794}{45} - \frac{2}{3}\zeta_2 - \frac{88}{5}\zeta_3\right)C_F^2\Bigg\} + \theta(N-3)(-1)^N \times \\
& \left[C_FC_A\left\{4S_{-3}(N-1) - 8S_{-3}(N+1) + \frac{136}{5}S_{-3}(N+2) - \frac{36}{5}S_{-3}(N+3) - 16S_{-3}(N) \right. \right. \\
& + \frac{4}{5}S_{-2}(N-2) - \left(\frac{136}{15} + 4\zeta_2\right)S_{-2}(N-1) - \left(\frac{376}{15} - 12\zeta_2\right)S_{-2}(N+1) - \frac{36}{5}S_{-2}(N+2) \\
& + \left(\frac{608}{15} - 8\zeta_2\right)S_{-2}(N) + \frac{2}{5}S_{-1}(N-3)\zeta_2 + \left(\frac{4}{5} - \frac{12}{5}\zeta_2\right)S_{-1}(N-2) \\
& - \left(\frac{737}{45} - 10\zeta_2 - 20\zeta_3\right)S_{-1}(N-1) + \left(\frac{2887}{45} + 2\zeta_2 - 60\zeta_3\right)S_{-1}(N+1) \\
& + \left(\frac{36}{5} - \frac{68}{5}\zeta_2\right)S_{-1}(N+2) + \frac{18}{5}S_{-1}(N+3)\zeta_2 - \left(\frac{502}{9} - 40\zeta_3\right)S_{-1}(N) \\
& - \left(\frac{50}{3} - 8\zeta_2\right)S_{-1,1}(N-1) + \left(\frac{142}{3} - 24\zeta_2\right)S_{-1,1}(N+1) - \left(\frac{92}{3} - 16\zeta_2\right)S_{-1,1}(N) \\
& - 8S_{-1,3}(N-1) + 24S_{-1,3}(N+1) - 16S_{-1,3}(N) - \frac{2}{5}S_1(N-3)\zeta_2 + \frac{12}{5}S_1(N-2)\zeta_2 \\
& - 14S_1(N-1)\zeta_2 + 6S_1(N+1)\zeta_2 - \frac{68}{5}S_1(N+2)\zeta_2 + \frac{18}{5}S_1(N+3)\zeta_2 + 16S_1(N)\zeta_2 \\
& - \frac{4}{5}S_{1,-2}(N-3) + \frac{24}{5}S_{1,-2}(N-2) - 28S_{1,-2}(N-1) + 12S_{1,-2}(N+1) \\
& - \frac{136}{5}S_{1,-2}(N+2) + \frac{36}{5}S_{1,-2}(N+3) + 32S_{1,-2}(N) + 4S_2(N-1)\zeta_2 - 12S_2(N+1)\zeta_2 \\
& \left. + 8S_2(N)\zeta_2 + 8S_{2,-2}(N-1) - 24S_{2,-2}(N+1) + 16S_{2,-2}(N)\right\} \\
& + C_Fn_f\left\{\frac{8}{3}S_{-2}(N-1) + \frac{8}{3}S_{-2}(N+1) - \frac{16}{3}S_{-2}(N) - \frac{14}{9}S_{-1}(N-1) - \frac{62}{9}S_{-1}(N+1) \right. \\
& \left. + \frac{76}{9}S_{-1}(N) - \frac{4}{3}S_{-1,1}(N-1) - \frac{4}{3}S_{-1,1}(N+1) + \frac{8}{3}S_{-1,1}(N)\right\} \\
& + C_F^2\left\{24S_{-3}(N+1) - \frac{272}{5}S_{-3}(N+2) + \frac{72}{5}S_{-3}(N+3) + 16S_{-3}(N) - \frac{8}{5}S_{-2}(N-2) \right. \\
& + \left(\frac{24}{5} + 8\zeta_2\right)S_{-2}(N-1) + \left(\frac{124}{5} - 24\zeta_2\right)S_{-2}(N+1) + \frac{72}{5}S_{-2}(N+2) \\
& - \left(\frac{212}{5} - 16\zeta_2\right)S_{-2}(N) - 12S_{-2,1}(N-1) - 12S_{-2,1}(N+1) + 24S_{-2,1}(N) \\
& - \frac{4}{5}S_{-1}(N-3)\zeta_2 - \left(\frac{8}{5} - \frac{24}{5}\zeta_2\right)S_{-1}(N-2) + \left(\frac{251}{5} - 20\zeta_2 - 40\zeta_3\right)S_{-1}(N-1) \\
& - \left(\frac{421}{5} + 4\zeta_2 - 120\zeta_3\right)S_{-1}(N+1) - \left(\frac{72}{5} - \frac{136}{5}\zeta_2\right)S_{-1}(N+2) - \frac{36}{5}S_{-1}(N+3)\zeta_2 \\
& \left. + (50 - 80\zeta_3)S_{-1}(N) + (42 - 16\zeta_2)S_{-1,1}(N-1) - (78 - 48\zeta_2)S_{-1,1}(N+1) \right\}
\end{aligned} \tag{199}$$

$$\begin{aligned}
& + \left(36 - 32\zeta_2 \right) S_{-1,1}(N) + 8S_{-1,1,1}(N-1) + 8S_{-1,1,1}(N+1) - 16S_{-1,1,1}(N) \\
& - 8S_{-1,2}(N-1) - 8S_{-1,2}(N+1) + 16S_{-1,2}(N) + 16S_{-1,3}(N-1) - 48S_{-1,3}(N+1) \\
& + 32S_{-1,3}(N) + \frac{4}{5}S_1(N-3)\zeta_2 - \frac{24}{5}S_1(N-2)\zeta_2 + 28S_1(N-1)\zeta_2 - 12S_1(N+1)\zeta_2 \\
& + \frac{136}{5}S_1(N+2)\zeta_2 - \frac{36}{5}S_1(N+3)\zeta_2 - 32S_1(N)\zeta_2 + \frac{8}{5}S_{1,-2}(N-3) - \frac{48}{5}S_{1,-2}(N-2) \\
& + 56S_{1,-2}(N-1) - 24S_{1,-2}(N+1) + \frac{272}{5}S_{1,-2}(N+2) - \frac{72}{5}S_{1,-2}(N+3) - 64S_{1,-2}(N) \\
& - 8S_2(N-1)\zeta_2 + 24S_2(N+1)\zeta_2 - 16S_2(N)\zeta_2 - 16S_{2,-2}(N-1) + 48S_{2,-2}(N+1) \\
& - 32S_{2,-2}(N) \Big\} \Big],
\end{aligned}$$

$$\begin{aligned}
c_{3,q}^{(2),-\text{ns}}(N) &= c_{2,q}^{(2),-\text{ns}}(N) + \delta(N-1) \left\{ \left(-\frac{9}{4} + 8\zeta_2 + \frac{64}{5}\zeta_2^2 - 38\zeta_3 \right) C_F \left(C_F - \frac{C_A}{2} \right) \right\} \\
&+ \delta(N-2) \left\{ \left(\frac{19}{270} + \frac{2}{3}\zeta_2 - \frac{8}{5}\zeta_3 \right) C_F \left(C_F - \frac{C_A}{2} \right) \right\} + \theta(N-3) (-1)^N \times \\
&\left[C_F \left(C_F - \frac{C_A}{2} \right) \left\{ 8S_{-3}(N-1) - 64S_{-3}(N+1) + \frac{128}{5}S_{-3}(N+2) + \frac{72}{5}S_{-3}(N+3) \right. \right. \\
&+ 16S_{-3}(N) - \frac{8}{5}S_{-2}(N-2) + \left(\frac{64}{5} + 8\zeta_2 \right) S_{-2}(N-1) + \left(\frac{224}{5} + 24\zeta_2 \right) S_{-2}(N+1) \\
&+ \frac{72}{5}S_{-2}(N+2) - \left(\frac{352}{5} + 32\zeta_2 \right) S_{-2}(N) - \frac{4}{5}S_{-1}(N-3)\zeta_2 - \left(\frac{8}{5} + \frac{16}{5}\zeta_2 \right) S_{-1}(N-2) \\
&- \left(\frac{304}{5} + 20\zeta_2 \right) S_{-1}(N-1) + \left(\frac{304}{5} + 28\zeta_2 \right) S_{-1}(N+1) - \left(\frac{72}{5} + \frac{64}{5}\zeta_2 \right) S_{-1}(N+2) \\
&- \frac{36}{5}S_{-1}(N+3)\zeta_2 + \left(16 + 16\zeta_2 \right) S_{-1}(N) + \frac{4}{5}S_1(N-3)\zeta_2 + \frac{16}{5}S_1(N-2)\zeta_2 \\
&+ \left(12\zeta_2 - 8\zeta_3 \right) S_1(N-1) + \left(36\zeta_2 - 24\zeta_3 \right) S_1(N+1) - \frac{64}{5}S_1(N+2)\zeta_2 \\
&- \frac{36}{5}S_1(N+3)\zeta_2 - \left(32\zeta_2 - 32\zeta_3 \right) S_1(N) - 16S_{1,-3}(N-1) - 48S_{1,-3}(N+1) \\
&+ 64S_{1,-3}(N) + \frac{8}{5}S_{1,-2}(N-3) + \frac{32}{5}S_{1,-2}(N-2) + 24S_{1,-2}(N-1) + 72S_{1,-2}(N+1) \\
&- \frac{128}{5}S_{1,-2}(N+2) - \frac{72}{5}S_{1,-2}(N+3) - 64S_{1,-2}(N) + 16S_{1,1}(N-1)\zeta_2 \\
&+ 48S_{1,1}(N+1)\zeta_2 - 64S_{1,1}(N)\zeta_2 + 32S_{1,1,-2}(N-1) + 96S_{1,1,-2}(N+1) \\
&- 128S_{1,1,-2}(N) - 8S_2(N-1)\zeta_2 - 24S_2(N+1)\zeta_2 + 32S_2(N)\zeta_2 - 16S_{2,-2}(N-1) \\
&- 48S_{2,-2}(N+1) + 64S_{2,-2}(N) \Big\} \Big],
\end{aligned}$$

$$\begin{aligned}
c_{2,q}^{(2),\text{ps}}(N) &= \delta(N-2) \left\{ -\frac{133}{81} C_F n_f \right\} + \theta(N-3) (-1)^N \times \\
&\left[C_F n_f \left\{ 20S_{-4}(N-1) + 20S_{-4}(N+1) - 40S_{-4}(N) + 2S_{-3}(N-1) - \frac{26}{3}S_{-3}(N+1) \right. \right. \\
&- \frac{64}{3}S_{-3}(N+2) + 28S_{-3}(N) - 16S_{-3,1}(N-1) - 16S_{-3,1}(N+1) + 32S_{-3,1}(N) \\
&+ 56S_{-2}(N-1) - \frac{392}{9}S_{-2}(N+1) + \frac{128}{9}S_{-2}(N+2) - \frac{80}{3}S_{-2}(N) - 16S_{-2,1}(N+1) \\
&+ 16S_{-2,1}(N+2) + 8S_{-2,1,1}(N-1) + 8S_{-2,1,1}(N+1) - 16S_{-2,1,1}(N) - 8S_{-2,2}(N-1)
\end{aligned}$$

$$\begin{aligned}
& -8S_{-2,2}(N+1) + 16S_{-2,2}(N) + \left(\frac{344}{27} - \frac{8}{3}\zeta_2\right)S_{-1}(N-2) - \left(\frac{818}{27} + \frac{16}{3}\zeta_2\right)S_{-1}(N-1) \\
& + \left(\frac{818}{27} + \frac{16}{3}\zeta_2\right)S_{-1}(N+1) + \left(\frac{448}{27} + \frac{8}{3}\zeta_2\right)S_{-1}(N+2) - \frac{88}{3}S_{-1}(N) \\
& - \frac{104}{9}S_{-1,1}(N-2) - \frac{208}{9}S_{-1,1}(N-1) + \frac{208}{9}S_{-1,1}(N+1) + \frac{32}{9}S_{-1,1}(N+2) \\
& + 8S_{-1,1}(N) + \frac{16}{3}S_{-1,1,1}(N-2) - \frac{28}{3}S_{-1,1,1}(N-1) + \frac{28}{3}S_{-1,1,1}(N+1) \\
& - \frac{16}{3}S_{-1,1,1}(N+2) - \frac{16}{3}S_{-1,2}(N-2) + \frac{28}{3}S_{-1,2}(N-1) - \frac{28}{3}S_{-1,2}(N+1) \\
& + \frac{16}{3}S_{-1,2}(N+2) - \frac{8}{3}S_1(N-2)\zeta_2 + \frac{32}{3}S_1(N-1)\zeta_2 + \frac{32}{3}S_1(N+1)\zeta_2 \\
& - \frac{8}{3}S_1(N+2)\zeta_2 - 16S_1(N)\zeta_2 - \frac{16}{3}S_{1,-2}(N-2) + \frac{64}{3}S_{1,-2}(N-1) + \frac{64}{3}S_{1,-2}(N+1) \\
& - \frac{16}{3}S_{1,-2}(N+2) - 32S_{1,-2}(N) \Bigg] ,
\end{aligned}$$

$$c_{2,g}^{(2)}(N) = \quad (202)$$

$$\begin{aligned}
& \delta(N-2) \left\{ -\left(\frac{4799}{810} - \frac{16}{5}\zeta_3\right)C_{Fn_f} + \left(\frac{115}{324} - 2\zeta_3\right)C_{An_f} \right\} + \theta(N-3)(-1)^N \times \\
& \left[C_{Fn_f} \left\{ -10S_{-4}(N-1) - 20S_{-4}(N+1) + 40S_{-4}(N+2) - 10S_{-4}(N) + 3S_{-3}(N-1) \right. \right. \\
& + \frac{172}{3}S_{-3}(N+1) - \frac{456}{5}S_{-3}(N+2) + \frac{96}{5}S_{-3}(N+3) + \frac{35}{3}S_{-3}(N) + 16S_{-3,1}(N-1) \\
& + 16S_{-3,1}(N+1) - 48S_{-3,1}(N+2) + 16S_{-3,1}(N) + \frac{8}{15}S_{-2}(N-2) \\
& - \left(\frac{244}{15} + 16\zeta_2\right)S_{-2}(N-1) - \left(\frac{103}{5} + 16\zeta_2\right)S_{-2}(N+1) + \left(\frac{216}{5} + 16\zeta_2\right)S_{-2}(N+2) \\
& - \left(\frac{103}{15} - 16\zeta_2\right)S_{-2}(N) - 16S_{-2,1}(N-1) - 16S_{-2,1}(N+1) + 72S_{-2,1}(N+2) \\
& - 40S_{-2,1}(N) - 16S_{-2,1,1}(N-1) - 8S_{-2,1,1}(N+1) + 40S_{-2,1,1}(N+2) - 16S_{-2,1,1}(N) \\
& + 12S_{-2,2}(N-1) + 8S_{-2,2}(N+1) - 32S_{-2,2}(N+2) + 12S_{-2,2}(N) + \frac{4}{15}S_{-1}(N-3)\zeta_2 \\
& + \left(\frac{8}{15} - \frac{4}{15}\zeta_2\right)S_{-1}(N-2) + \left(\frac{213}{5} + 24\zeta_2 + 16\zeta_3\right)S_{-1}(N-1) \\
& - \left(\frac{203}{5} + \frac{32}{3}\zeta_2 + 40\zeta_3\right)S_{-1}(N+1) - \left(\frac{36}{5} - \frac{48}{5}\zeta_2 - 8\zeta_3\right)S_{-1}(N+2) - \frac{48}{5}S_{-1}(N+3)\zeta_2 \\
& + \left(\frac{14}{3} - \frac{40}{3}\zeta_2 + 16\zeta_3\right)S_{-1}(N) + \left(14 + 16\zeta_2\right)S_{-1,1}(N-1) - \left(16 + 16\zeta_2\right)S_{-1,1}(N+1) \\
& - \left(24 + 16\zeta_2\right)S_{-1,1}(N+2) + \left(26 + 16\zeta_2\right)S_{-1,1}(N) + 26S_{-1,1,1}(N-1) \\
& - 8S_{-1,1,1}(N+1) - 72S_{-1,1,1}(N+2) + 54S_{-1,1,1}(N) + 20S_{-1,1,1,1}(N-1) \\
& - 40S_{-1,1,1,1}(N+2) + 20S_{-1,1,1,1}(N) - 16S_{-1,1,2}(N-1) + 32S_{-1,1,2}(N+2) \\
& - 16S_{-1,1,2}(N) - 26S_{-1,2}(N-1) + 8S_{-1,2}(N+1) + 72S_{-1,2}(N+2) - 54S_{-1,2}(N) \\
& - 24S_{-1,2,1}(N-1) + 48S_{-1,2,1}(N+2) - 24S_{-1,2,1}(N) + 4S_{-1,3}(N-1) \\
& + 16S_{-1,3}(N+1) - 24S_{-1,3}(N+2) + 4S_{-1,3}(N) - \frac{4}{15}S_1(N-3)\zeta_2 + \frac{4}{15}S_1(N-2)\zeta_2 \\
& - \left(24\zeta_2 - 8\zeta_3\right)S_1(N-1) - \left(\frac{32}{3}\zeta_2 - 24\zeta_3\right)S_1(N+1) + \left(\frac{48}{5}\zeta_2 - 8\zeta_3\right)S_1(N+2)
\end{aligned}$$

$$\begin{aligned}
& -\frac{48}{5}S_1(N+3)\zeta_2 + \left(\frac{104}{3}\zeta_2 - 24\zeta_3\right)S_1(N) + 16S_{1,-3}(N-1) + 48S_{1,-3}(N+1) \\
& -16S_{1,-3}(N+2) - 48S_{1,-3}(N) - \frac{8}{15}S_{1,-2}(N-3) + \frac{8}{15}S_{1,-2}(N-2) - 48S_{1,-2}(N-1) \\
& -\frac{64}{3}S_{1,-2}(N+1) + \frac{96}{5}S_{1,-2}(N+2) - \frac{96}{5}S_{1,-2}(N+3) + \frac{208}{3}S_{1,-2}(N) \\
& -16S_{1,1}(N-1)\zeta_2 - 48S_{1,1}(N+1)\zeta_2 + 16S_{1,1}(N+2)\zeta_2 + 48S_{1,1}(N)\zeta_2 \\
& -32S_{1,1,-2}(N-1) - 96S_{1,1,-2}(N+1) + 32S_{1,1,-2}(N+2) + 96S_{1,1,-2}(N) \\
& +16S_2(N-1)\zeta_2 + 16S_2(N+1)\zeta_2 - 16S_2(N+2)\zeta_2 - 16S_2(N)\zeta_2 + 32S_{2,-2}(N-1) \\
& +32S_{2,-2}(N+1) - 32S_{2,-2}(N+2) - 32S_{2,-2}(N) \Big\} \\
& + C_{Anf} \Big\{ 20S_{-4}(N-1) + 56S_{-4}(N+1) - 76S_{-4}(N) + 2S_{-3}(N-1) - \frac{140}{3}S_{-3}(N+1) \\
& -\frac{388}{3}S_{-3}(N+2) + 174S_{-3}(N) - 8S_{-3,1}(N-1) - 48S_{-3,1}(N+1) - 16S_{-3,1}(N+2) \\
& +72S_{-3,1}(N) + 58S_{-2}(N-1) - \left(\frac{338}{9} + 8\zeta_2\right)S_{-2}(N+1) + \left(\frac{2090}{9} + 8\zeta_2\right)S_{-2}(N+2) \\
& -\frac{758}{3}S_{-2}(N) - 8S_{-2,1}(N-1) - 4S_{-2,1}(N+1) + 148S_{-2,1}(N+2) - 136S_{-2,1}(N) \\
& +32S_{-2,1,1}(N+1) + 16S_{-2,1,1}(N+2) - 48S_{-2,1,1}(N) - 32S_{-2,2}(N+1) \\
& -16S_{-2,2}(N+2) + 48S_{-2,2}(N) + \left(\frac{344}{27} - \frac{8}{3}\zeta_2\right)S_{-1}(N-2) \\
& -\left(\frac{1061}{27} + \frac{28}{3}\zeta_2 + 14\zeta_3\right)S_{-1}(N-1) + \left(\frac{1277}{27} + \frac{40}{3}\zeta_2 + 20\zeta_3\right)S_{-1}(N+1) \\
& -\left(\frac{4493}{27} + \frac{40}{3}\zeta_2 - 8\zeta_3\right)S_{-1}(N+2) + \left(\frac{437}{3} + 12\zeta_2 - 14\zeta_3\right)S_{-1}(N) - \frac{104}{9}S_{-1,1}(N-2) \\
& -\left(\frac{82}{9} + 4\zeta_2\right)S_{-1,1}(N-1) + \left(\frac{208}{9} + 8\zeta_2\right)S_{-1,1}(N+1) - \frac{1570}{9}S_{-1,1}(N+2) \\
& +\left(172 - 4\zeta_2\right)S_{-1,1}(N) + \frac{16}{3}S_{-1,1,1}(N-2) - \frac{4}{3}S_{-1,1,1}(N-1) + \frac{28}{3}S_{-1,1,1}(N+1) \\
& -\frac{244}{3}S_{-1,1,1}(N+2) + 68S_{-1,1,1}(N) + 4S_{-1,1,1,1}(N-1) - 8S_{-1,1,1,1}(N+2) \\
& +4S_{-1,1,1,1}(N) - 12S_{-1,1,2}(N-1) + 24S_{-1,1,2}(N+2) - 12S_{-1,1,2}(N) - \frac{16}{3}S_{-1,2}(N-2) \\
& +\frac{4}{3}S_{-1,2}(N-1) - \frac{28}{3}S_{-1,2}(N+1) + \frac{268}{3}S_{-1,2}(N+2) - 76S_{-1,2}(N) - 4S_{-1,2,1}(N-1) \\
& +8S_{-1,2,1}(N+2) - 4S_{-1,2,1}(N) + 12S_{-1,3}(N-1) - 8S_{-1,3}(N+1) - 16S_{-1,3}(N+2) \\
& +12S_{-1,3}(N) - \frac{8}{3}S_1(N-2)\zeta_2 + \left(\frac{44}{3}\zeta_2 + 2\zeta_3\right)S_1(N-1) - \left(\frac{40}{3}\zeta_2 - 12\zeta_3\right)S_1(N+1) \\
& +\left(\frac{40}{3}\zeta_2 - 8\zeta_3\right)S_1(N+2) - \left(12\zeta_2 + 6\zeta_3\right)S_1(N) + 8S_{1,-3}(N-1) + 40S_{1,-3}(N+1) \\
& -24S_{1,-3}(N+2) - 24S_{1,-3}(N) - \frac{16}{3}S_{1,-2}(N-2) + \frac{88}{3}S_{1,-2}(N-1) - \frac{80}{3}S_{1,-2}(N+1) \\
& +\frac{80}{3}S_{1,-2}(N+2) - 24S_{1,-2}(N) - 8S_{1,-2,1}(N-1) - 32S_{1,-2,1}(N+1) \\
& +16S_{1,-2,1}(N+2) + 24S_{1,-2,1}(N) + 4S_{1,1}(N-1)\zeta_2 + 8S_{1,1}(N+1)\zeta_2 - 12S_{1,1}(N)\zeta_2 \\
& +8S_{1,1,-2}(N-1) + 16S_{1,1,-2}(N+1) - 24S_{1,1,-2}(N) + 8S_2(N+1)\zeta_2 - 8S_2(N+2)\zeta_2 \\
& +16S_{2,-2}(N+1) - 16S_{2,-2}(N+2) \Big\} ,
\end{aligned}$$

$$\begin{aligned}
c_{L,q}^{(2),\text{ns}}(N) = & \delta(N-2) \left\{ -\frac{92}{27} C_F n_f - \left(\frac{5756}{135} - \frac{64}{5} \zeta_3 \right) C_F \left(C_F - \frac{C_A}{2} \right) + \frac{770}{27} C_F^2 \right\} + \theta(N-3) (-1)^N \times \\
& \left[C_F n_f \left\{ -\frac{16}{3} S_{-2}(N+1) + \frac{16}{3} S_{-2}(N) - \frac{8}{3} S_{-1}(N-1) + \frac{100}{9} S_{-1}(N+1) - \frac{76}{9} S_{-1}(N) \right. \right. \\
& \quad \left. \left. + \frac{8}{3} S_{-1,1}(N+1) - \frac{8}{3} S_{-1,1}(N) \right\} \right. \\
& + C_F \left(C_F - \frac{C_A}{2} \right) \left\{ 32 S_{-3}(N+1) + \frac{96}{5} S_{-3}(N+2) - \frac{96}{5} S_{-3}(N+3) - 32 S_{-3}(N) \right. \\
& \quad + \frac{64}{5} S_{-2}(N-2) - \frac{32}{5} S_{-2}(N-1) - \frac{1216}{15} S_{-2}(N+1) - \frac{96}{5} S_{-2}(N+2) + \frac{1408}{15} S_{-2}(N) \\
& \quad + \frac{32}{5} S_{-1}(N-3) \zeta_2 + \left(\frac{64}{5} - \frac{32}{5} \zeta_2 \right) S_{-1}(N-2) - \left(\frac{584}{15} - 32 \zeta_2 \right) S_{-1}(N-1) \\
& \quad + \left(\frac{6052}{45} - 16 \zeta_2 - 80 \zeta_3 \right) S_{-1}(N+1) + \left(\frac{96}{5} - \frac{48}{5} \zeta_2 \right) S_{-1}(N+2) + \frac{48}{5} S_{-1}(N+3) \zeta_2 \\
& \quad - \left(\frac{1148}{9} + 16 \zeta_2 - 80 \zeta_3 \right) S_{-1}(N) + \left(\frac{184}{3} - 32 \zeta_2 \right) S_{-1,1}(N+1) - \left(\frac{184}{3} - 32 \zeta_2 \right) S_{-1,1}(N) \\
& \quad + 32 S_{-1,3}(N+1) - 32 S_{-1,3}(N) - \frac{32}{5} S_1(N-3) \zeta_2 + \frac{32}{5} S_1(N-2) \zeta_2 - 32 S_1(N-1) \zeta_2 \\
& \quad - \left(16 \zeta_2 - 16 \zeta_3 \right) S_1(N+1) - \frac{48}{5} S_1(N+2) \zeta_2 + \frac{48}{5} S_1(N+3) \zeta_2 + \left(48 \zeta_2 - 16 \zeta_3 \right) S_1(N) \\
& \quad + 32 S_{1,-3}(N+1) - 32 S_{1,-3}(N) - \frac{64}{5} S_{1,-2}(N-3) + \frac{64}{5} S_{1,-2}(N-2) - 64 S_{1,-2}(N-1) \\
& \quad - 32 S_{1,-2}(N+1) - \frac{96}{5} S_{1,-2}(N+2) + \frac{96}{5} S_{1,-2}(N+3) + 96 S_{1,-2}(N) - 32 S_{1,1}(N+1) \zeta_2 \\
& \quad \left. \left. + 32 S_{1,1}(N) \zeta_2 - 64 S_{1,1,-2}(N+1) + 64 S_{1,1,-2}(N) \right\} \right. \\
& + C_F^2 \left\{ -16 S_{-3}(N+1) + 16 S_{-3}(N) - 8 S_{-2}(N-1) + \frac{176}{3} S_{-2}(N+1) - \frac{152}{3} S_{-2}(N) \right. \\
& \quad + 24 S_{-2,1}(N+1) - 24 S_{-2,1}(N) + \frac{52}{3} S_{-1}(N-1) - \frac{710}{9} S_{-1}(N+1) + \frac{554}{9} S_{-1}(N) \\
& \quad + 8 S_{-1,1}(N-1) - \frac{100}{3} S_{-1,1}(N+1) + \frac{76}{3} S_{-1,1}(N) - 16 S_{-1,1,1}(N+1) + 16 S_{-1,1,1}(N) \\
& \quad \left. \left. + 16 S_{-1,2}(N+1) - 16 S_{-1,2}(N) \right\} \right], \\
\end{aligned}$$

$$\begin{aligned}
c_{L,q}^{(2),\text{ps}}(N) = & \delta(N-2) \left\{ -\frac{80}{27} C_F n_f \right\} + \theta(N-3) (-1)^N \times \\
& \left[C_F n_f \left\{ -32 S_{-3}(N+1) + 32 S_{-3}(N) + 16 S_{-2}(N-1) - 48 S_{-2}(N+1) + 32 S_{-2}(N+2) \right. \right. \\
& \quad + 16 S_{-2,1}(N+1) - 16 S_{-2,1}(N) - \frac{16}{9} S_{-1}(N-2) - \frac{32}{9} S_{-1}(N-1) + \frac{32}{9} S_{-1}(N+1) \\
& \quad + \frac{160}{9} S_{-1}(N+2) - 16 S_{-1}(N) - \frac{16}{3} S_{-1,1}(N-2) - \frac{32}{3} S_{-1,1}(N-1) + \frac{32}{3} S_{-1,1}(N+1) \\
& \quad \left. \left. - \frac{32}{3} S_{-1,1}(N+2) + 16 S_{-1,1}(N) \right\} \right], \\
\end{aligned}$$

$$\begin{aligned}
c_{L,g}^{(2)}(N) = & \delta(N-2) \left\{ -\left(\frac{116}{135} + \frac{16}{5}\zeta_3\right) C_{Fnf} + \frac{173}{27} C_{Anf} \right\} + \theta(N-3) (-1)^N \times \\
& \left[C_{Fnf} \left\{ \frac{64}{3} S_{-3}(N+1) - \frac{64}{5} S_{-3}(N+2) + \frac{64}{5} S_{-3}(N+3) - \frac{64}{3} S_{-3}(N) + \frac{32}{15} S_{-2}(N-2) \right. \right. \\
& - \frac{136}{15} S_{-2}(N-1) - \frac{112}{5} S_{-2}(N+1) - \frac{96}{5} S_{-2}(N+2) + \frac{728}{15} S_{-2}(N) \\
& - 16 S_{-2,1}(N+1) + 16 S_{-2,1}(N) + \frac{16}{15} S_{-1}(N-3) \zeta_2 + \left(\frac{32}{15} - \frac{16}{15} \zeta_2\right) S_{-1}(N-2) \\
& + \frac{32}{5} S_{-1}(N-1) - \left(\frac{32}{5} - \frac{16}{3} \zeta_2\right) S_{-1}(N+1) + \left(\frac{336}{5} + \frac{32}{5} \zeta_2\right) S_{-1}(N+2) \\
& - \frac{32}{5} S_{-1}(N+3) \zeta_2 - \left(\frac{208}{3} + \frac{16}{3} \zeta_2\right) S_{-1}(N) + 8 S_{-1,1}(N-1) - 8 S_{-1,1}(N+1) \\
& + 32 S_{-1,1}(N+2) - 32 S_{-1,1}(N) - \frac{16}{15} S_1(N-3) \zeta_2 + \frac{16}{15} S_1(N-2) \zeta_2 + \frac{16}{3} S_1(N+1) \zeta_2 \\
& + \frac{32}{5} S_1(N+2) \zeta_2 - \frac{32}{5} S_1(N+3) \zeta_2 - \frac{16}{3} S_1(N) \zeta_2 - \frac{32}{15} S_{1,-2}(N-3) + \frac{32}{15} S_{1,-2}(N-2) \\
& \left. + \frac{32}{3} S_{1,-2}(N+1) + \frac{64}{5} S_{1,-2}(N+2) - \frac{64}{5} S_{1,-2}(N+3) - \frac{32}{3} S_{1,-2}(N) \right\} \\
& + C_{Anf} \left\{ -96 S_{-3}(N+1) + 96 S_{-3}(N) + 16 S_{-2}(N-1) - 80 S_{-2}(N+1) + 208 S_{-2}(N+2) \right. \\
& - 144 S_{-2}(N) + 64 S_{-2,1}(N+1) + 32 S_{-2,1}(N+2) - 96 S_{-2,1}(N) - \frac{16}{9} S_{-1}(N-2) \\
& - \frac{32}{9} S_{-1}(N-1) + \frac{32}{9} S_{-1}(N+1) - \left(\frac{848}{9} + 16 \zeta_2\right) S_{-1}(N+2) + \left(96 + 16 \zeta_2\right) S_{-1}(N) \\
& - \frac{16}{3} S_{-1,1}(N-2) - \frac{32}{3} S_{-1,1}(N-1) + \frac{32}{3} S_{-1,1}(N+1) - \frac{464}{3} S_{-1,1}(N+2) \\
& + 160 S_{-1,1}(N) - 32 S_{-1,1,1}(N+2) + 32 S_{-1,1,1}(N) + 32 S_{-1,2}(N+2) - 32 S_{-1,2}(N) \\
& - 32 S_1(N+1) \zeta_2 + 16 S_1(N+2) \zeta_2 + 16 S_1(N) \zeta_2 - 64 S_{1,-2}(N+1) + 32 S_{1,-2}(N+2) \\
& \left. + 32 S_{1,-2}(N) \right\} \Big].
\end{aligned} \tag{205}$$

Appendix C

Here we present the x -space expressions of the 2-loop coefficient functions. We find,³

$$\begin{aligned}
c_{2,q}^{(2),+ns}(x) = & \\
& C_{FCA} \left\{ -\frac{3229}{180} - \frac{4}{5x} + \frac{2191}{20}x - \frac{36}{5}x^2 + 4p_{qq}(x) \left(-\frac{3155}{432} + \frac{11}{3}\zeta_2 + \frac{1}{2}\zeta_3 - 3H_{-2,0}(x) \right. \right. \\
& - \frac{239}{24}H_0(x) + H_0(x)\zeta_2 - \frac{55}{12}H_{0,0}(x) - \frac{3}{2}H_{0,0,0}(x) - \frac{367}{72}H_1(x) + 3H_1(x)\zeta_2 \\
& - \frac{11}{6}H_{1,0}(x) - 2H_{1,0,0}(x) - \frac{11}{6}H_{1,1}(x) + H_{1,1,0}(x) - H_{1,2}(x) - \frac{11}{3}H_2(x) - H_3(x) \Big) \\
& - \frac{1}{6}(133 - 371x)H_1(x) - 4\left(5 + \frac{1}{5x^2} + x - 6x^2 - \frac{9}{5}x^3\right)H_{-1,0}(x) + \frac{1}{30}\left(13 + \frac{24}{x} \right. \\
& \left. \left. + 1753x - 216x^2\right)H_0(x) + 4(1 - 5x)\left(\zeta_3 + H_1(x)\zeta_2 - H_{-2,0}(x) - H_{1,0,0}(x)\right) \right\}
\end{aligned} \tag{206}$$

³ The expression for the coefficient function $c_{2,q}^{(2),ps}$ in eq.(13) of ref.[13] contains a typographical error. The term $+488/27x^2$ should read $+448/27x^2$.

$$\begin{aligned}
& +2\left(1-3x+12x^2+\frac{18}{5}x^3\right)\zeta_2-4\left(1-x+6x^2+\frac{9}{5}x^3\right)H_{0,0}(x)-4(1+x)H_2(x) \\
& -\left(\frac{5465}{72}+\frac{251}{3}\zeta_2-\frac{71}{5}\zeta_2^2-\frac{140}{3}\zeta_3\right)\delta(1-x)\Big\} \\
& + C_F n_f \left\{-\frac{23}{18}-\frac{27}{2}x+\frac{4}{3}p_{\text{qq}}(x)\left(\frac{247}{72}-2\zeta_2+\frac{19}{4}H_0(x)+\frac{5}{2}H_{0,0}(x)+\frac{29}{12}H_1(x)+H_{1,0}(x)\right.\right. \\
& \quad \left.+H_{1,1}(x)+2H_2(x)\right)-\frac{1}{3}(7+19x)H_0(x)-\frac{1}{3}(1+13x)H_1(x) \\
& \quad \left.+\left(\frac{457}{36}+\frac{38}{3}\zeta_2+\frac{4}{3}\zeta_3\right)\delta(1-x)\right\} \\
& + C_F^2 \left\{\frac{407}{20}+\frac{8}{5x}-\frac{1917}{20}x+\frac{72}{5}x^2+4p_{\text{qq}}(x)\left(\frac{51}{16}+3\zeta_2+8\zeta_3+6H_{-2,0}(x)+\frac{61}{8}H_0(x)\right.\right. \\
& \quad \left.+6H_0(x)\zeta_2-\frac{3}{4}H_{0,0}(x)-2H_{0,0,0}(x)+\frac{27}{8}H_1(x)+2H_1(x)\zeta_2-\frac{9}{2}H_{1,0}(x)\right. \\
& \quad \left.-3H_{1,0,0}(x)-\frac{9}{2}H_{1,1}(x)-8H_{1,1,0}(x)-6H_{1,1,1}(x)-6H_{1,2}(x)-3H_2(x)-6H_{2,0}(x)\right. \\
& \quad \left.-7H_{2,1}(x)-6H_3(x)\right)+\left(\frac{13}{10}-\frac{8}{5x}-\frac{407}{10}x+\frac{72}{5}x^2\right)H_0(x)+\left(\frac{91}{2}-\frac{141}{2}x\right)H_1(x) \\
& \quad +8\left(5+\frac{1}{5x^2}+x-6x^2-\frac{9}{5}x^3\right)H_{-1,0}(x)+\left(29+25x+48x^2+\frac{72}{5}x^3\right)H_{0,0}(x) \\
& \quad -12(1-3x)\zeta_3+8(1-5x)\left(H_{-2,0}(x)+H_{1,0,0}(x)-H_1(x)\zeta_2\right) \\
& \quad +2(1+x)\left(5H_{0,0,0}(x)-4H_0(x)\zeta_2+7H_{1,0}(x)+2H_{2,0}(x)+2H_{2,1}(x)+4H_3(x)\right) \\
& \quad -24\left(1+x+2x^2+\frac{3}{5}x^3\right)\zeta_2+4(7+11x)H_2(x)+2(5+9x)H_{1,1}(x) \\
& \quad \left.+\left(\frac{331}{8}+69\zeta_2+6\zeta_2^2-78\zeta_3\right)\delta(1-x)\right\},
\end{aligned}$$

$$c_{2,\text{q}}^{(2),-\text{ns}}(x) = \quad (207)$$

$$\begin{aligned}
& C_F\left(C_F-\frac{C_A}{2}\right)\left\{-\frac{162}{5}+\frac{8}{5x}+\frac{82}{5}x+\frac{72}{5}x^2+4p_{\text{qq}}(-x)\left(7\zeta_3+6H_{-2,0}(x)-8H_{-1}(x)\zeta_2\right.\right. \\
& \quad \left.-8H_{-1,-1,0}(x)+10H_{-1,0,0}(x)+4H_{-1,2}(x)-2H_0(x)+2H_0(x)\zeta_2-3H_{0,0,0}(x)\right. \\
& \quad \left.-2H_3(x)\right)+32\left(1+\frac{1}{20x^2}+x+\frac{3}{2}x^2-\frac{9}{20}x^3\right)H_{-1,0}(x)-16(1-x)H_1(x) \\
& \quad -\frac{2}{5}\left(13+\frac{4}{x}+53x-36x^2\right)H_0(x)-8(1+x)H_2(x)-8\left(1+4x+6x^2-\frac{9}{5}x^3\right)H_{0,0}(x) \\
& \quad +8(1+5x)\left(\zeta_3+H_{-2,0}(x)-2H_{-1,-1,0}(x)+H_{-1,0,0}(x)-H_{-1}(x)\zeta_2\right) \\
& \quad \left.+4\left(1+7x+12x^2-\frac{18}{5}x^3\right)\zeta_2\right\},
\end{aligned}$$

$$c_{3,\text{q}}^{(2),+\text{ns}}(x) = c_{2,\text{q}}^{(2),+\text{ns}}(x) + \quad (208)$$

$$\begin{aligned}
& C_F C_A \left\{\frac{701}{45}+\frac{4}{5x}-\frac{3211}{45}x+\frac{36}{5}x^2+\frac{2}{3}(25-71x)H_1(x)-8(1-3x)\left(\zeta_3+H_1(x)\zeta_2\right.\right. \\
& \quad \left.-H_{-2,0}(x)-H_{1,0,0}(x)\right)+4\left(6+\frac{1}{5x^2}+\frac{1}{x}+2x-5x^2-\frac{9}{5}x^3\right)H_{-1,0}(x)
\end{aligned}$$

$$\begin{aligned}
& -\frac{4}{15}\left(31 + \frac{3}{x} + 121x - 27x^2\right)H_0(x) + 4\left(1 + 3x - 5x^2 - \frac{9}{5}x^3\right)\left(\zeta_2 - H_{0,0}(x)\right)\Big\} \\
& + C_F n_f \left\{ \frac{14}{9} + \frac{62}{9}x + \frac{4}{3}(1+x)\left(2H_0(x) + H_1(x)\right) \right\} \\
& + C_F^2 \left\{ -\frac{243}{5} - \frac{8}{5x} + \frac{493}{5}x - \frac{72}{5}x^2 - 2(21 - 39x)H_1(x) + \frac{8}{5}\left(2 + \frac{1}{x} + \frac{49}{2}x - 9x^2\right)H_0(x) \right. \\
& \quad - 8\left(6 + \frac{1}{5x^2} + \frac{1}{x} + 2x - 5x^2 - \frac{9}{5}x^3\right)H_{-1,0}(x) + 4\left(1 - 3x + 10x^2 + \frac{18}{5}x^3\right)\zeta_2 \\
& \quad + 16(1 - 3x)\left(\zeta_3 + H_1(x)\zeta_2 - H_{-2,0}(x) - H_{1,0,0}(x)\right) \\
& \quad \left. - 8(1+x)\left(H_{1,0}(x) + H_{1,1}(x) + \frac{3}{2}H_2(x)\right) + 8\left(2x - 5x^2 - \frac{9}{5}x^3\right)H_{0,0}(x) \right\},
\end{aligned}$$

$$\begin{aligned}
c_{3,q}^{(2),-\text{ns}}(x) &= c_{2,q}^{(2),-\text{ns}}(x) + \\
& C_F \left(C_F - \frac{C_A}{2} \right) \left\{ \frac{312}{5} - \frac{8}{5x} - \frac{232}{5}x - \frac{72}{5}x^2 - 8\left(4 + \frac{1}{5x^2} - \frac{1}{x} + 4x + 5x^2 - \frac{9}{5}x^3\right)H_{-1,0}(x) \right. \\
& \quad + \frac{8}{5}\left(7 + \frac{1}{x} + 37x - 9x^2\right)H_0(x) + 8\left(1 - 3x - 5x^2 + \frac{9}{5}x^3\right)\left(\zeta_2 - H_{0,0}(x)\right) \\
& \quad \left. - 16(1 + 3x)\left(\zeta_3 - H_{-1}(x)\zeta_2 + H_{-2,0}(x) - 2H_{-1,-1,0}(x) + H_{-1,0,0}(x)\right) \right\},
\end{aligned} \tag{209}$$

$$\begin{aligned}
c_{2,q}^{(2),\text{ps}}(x) &= \\
& n_f C_F \left\{ \frac{158}{9} + \frac{344}{27x} - \frac{422}{9}x + \frac{448}{27}x^2 + \frac{8}{3}\left(13 - \frac{13}{3x} - 10x + \frac{4}{3}x^2\right)H_1(x) \right. \\
& \quad - 16\left(1 + \frac{1}{3x} + x + \frac{1}{3}x^2\right)H_{-1,0}(x) + 4\left(1 + \frac{4}{3x} - x - \frac{4}{3}x^2\right)\left(H_{1,0}(x) + H_{1,1}(x)\right) \\
& \quad - 16x^2H_2(x) - 2\left(1 - 15x + \frac{32}{3}x^2\right)H_{0,0}(x) + 56\left(1 - \frac{11}{21}x - \frac{16}{63}x^2\right)H_0(x) \\
& \quad - 8(1+x)\left(\zeta_3 + 2H_0(x)\zeta_2 - \frac{5}{2}H_{0,0,0}(x) - H_{2,0}(x) - H_{2,1}(x) - 2H_3(x)\right) \\
& \quad \left. - \frac{16}{3}\left(\frac{1}{x} + 3x - 3x^2\right)\zeta_2 \right\},
\end{aligned} \tag{210}$$

$$\begin{aligned}
c_{2,g}^{(2)}(x) &= \\
& n_f C_F \left\{ -\frac{647}{15} + \frac{8}{15x} + \frac{239}{5}x - \frac{36}{5}x^2 + 48\left(1 + \frac{1}{90x^2} + \frac{4}{9}x + \frac{2}{5}x^3\right)H_{-1,0}(x) \right. \\
& \quad - \frac{4}{15}\left(59 + \frac{2}{x} - \frac{339}{4}x + 162x^2\right)H_0(x) - 3\left(1 - \frac{44}{9}x + 24x^2 + \frac{32}{5}x^3\right)H_{0,0}(x) \\
& \quad - 8\left(2 - 7x + 9x^2\right)H_2(x) - 2\left(13 - 40x + 36x^2\right)\left(H_{1,0}(x) + H_{1,1}(x)\right) \\
& \quad - 2\left(7 - 20x + 12x^2\right)H_1(x) + 16\left(1 - \frac{13}{6}x + \frac{9}{2}x^2 + \frac{6}{5}x^3\right)\zeta_2 + 8(4 + 9x^2)\zeta_3 \\
& \quad + 32(1 + x^2)H_{-2,0}(x) + 16x^2\left(H_{-1}(x)\zeta_2 + 2H_{-1,-1,0}(x) - H_{-1,0,0}(x) + H_0(x)\zeta_2 \right. \\
& \quad \left. - \frac{5}{4}H_{0,0,0}(x) + H_1(x)\zeta_2 - H_{1,0,0}(x) - \frac{1}{2}H_{2,0}(x) - \frac{1}{2}H_{2,1}(x) - H_3(x)\right) \\
& \quad \left. - 16\left(H_{-1}(x)\zeta_2 + 2H_{-1,-1,0}(x) - H_{-1,0,0}(x)\right)p_{\text{qg}}(-x) + 16\left(H_0(x)\zeta_2 - \frac{5}{8}H_{0,0,0}(x)\right) \right\}
\end{aligned} \tag{211}$$

$$\begin{aligned}
& + \frac{1}{2}H_1(x)\zeta_2 - \frac{1}{4}H_{1,0,0}(x) - H_{1,1,0}(x) - \frac{5}{4}H_{1,1,1}(x) - \frac{3}{2}H_{1,2}(x) - \frac{3}{4}H_{2,0}(x) \\
& - H_{2,1}(x) - H_3(x) \Big) p_{\text{qg}}(x) \Big\} \\
& + n_f C_A \Big\{ \frac{239}{9} + \frac{344}{27x} + \frac{1072}{9}x - \frac{4493}{27}x^2 - 4 \Big(1 - \frac{4}{3x} - 20x + \frac{67}{3}x^2 \Big) H_{1,0}(x) \\
& - 4 \Big(1 - \frac{4}{3x} - 18x + \frac{61}{3}x^2 \Big) H_{1,1}(x) + 8 \Big(1 - \frac{2}{3x} - 18x + \frac{37}{2}x^2 \Big) \zeta_2 \\
& + \frac{2}{3} \Big(31 - \frac{52}{3x} + 227x - \frac{785}{3}x^2 \Big) H_1(x) - 24 \Big(1 + \frac{2}{9x} - \frac{10}{9}x^2 \Big) H_{-1,0}(x) \\
& - 2 \Big(1 - 88x + \frac{194}{3}x^2 \Big) H_{0,0}(x) - 8 \Big(1 - 18x + \frac{37}{2}x^2 \Big) H_2(x) + 4 \Big(1 - 12x + 6x^2 \Big) \zeta_3 \\
& + 4(5 + 14x)H_{0,0,0}(x) + 58 \Big(1 + \frac{292}{87}x - \frac{1045}{261}x^2 \Big) H_0(x) \\
& - 8 \Big(1 + 8x - 2x^2 \Big) \Big(H_0(x)\zeta_2 - H_3(x) \Big) + 16x(3 - x) \Big(H_{2,0}(x) + H_{2,1}(x) \Big) \\
& + 16x^2 \Big(H_{-2,0}(x) - \frac{1}{2}H_{-1}(x)\zeta_2 - H_{-1,-1,0}(x) + \frac{1}{2}H_{-1,0,0}(x) - \frac{1}{2}H_1(x)\zeta_2 \\
& + \frac{1}{2}H_{1,0,0}(x) \Big) - 4 \Big(H_{-1}(x)\zeta_2 - 2H_{-1,-1,0}(x) - 2H_{-1,0,0}(x) - 2H_{-1,2}(x) \Big) p_{\text{qg}}(-x) \\
& + 8 \Big(H_1(x)\zeta_2 - \frac{3}{2}H_{1,0,0}(x) - \frac{3}{2}H_{1,1,0}(x) - \frac{1}{2}H_{1,1,1}(x) - \frac{1}{2}H_{1,2}(x) \Big) p_{\text{qg}}(x) \Big\} ,
\end{aligned}$$

$$c_{L,q}^{(2),\text{ns}}(x) = \quad (212)$$

$$\begin{aligned}
& n_f C_F \Big\{ \frac{8}{3} - \frac{100}{9}x - \frac{8}{3}x \Big(2H_0(x) + H_1(x) \Big) \Big\} \\
& + C_F \Big(C_F - \frac{C_A}{2} \Big) \Big\{ \frac{392}{15} + \frac{64}{5x} - \frac{6916}{45}x + \frac{96}{5}x^2 + 32 \Big(2 + \frac{2}{5x^2} + x - \frac{3}{5}x^3 \Big) H_{-1,0}(x) \\
& + \frac{32}{5} \Big(1 - \frac{2}{x} - \frac{47}{3}x + 3x^2 \Big) H_0(x) - \frac{184}{3}x H_1(x) + 32x \Big(1 - \frac{3}{5}x^2 \Big) \Big(\zeta_2 - H_{0,0}(x) \Big) \\
& + 32x \Big(2\zeta_3 + \zeta_2 H_1(x) - \zeta_2 H_{-1}(x) - 2H_{-1,-1,0}(x) + H_{-1,0,0}(x) - H_{1,0,0}(x) \Big) \Big\} \\
& + C_F^2 \Big\{ -\frac{52}{3} + \frac{710}{9}x - 8 \Big(1 - \frac{22}{3}x \Big) H_0(x) - 8 \Big(1 - \frac{25}{6}x \Big) H_1(x) - 16x \Big(\frac{3}{2}\zeta_2 - H_{0,0}(x) \\
& - H_{1,0}(x) - H_{1,1}(x) - \frac{3}{2}H_2(x) \Big) \Big\} ,
\end{aligned}$$

$$c_{L,q}^{(2),\text{ps}}(x) = \quad (213)$$

$$\begin{aligned}
& n_f C_F \Big\{ \frac{16}{3} - \frac{16}{9x} - \frac{64}{3}x + \frac{160}{9}x^2 + \frac{16}{3} \Big(3 - \frac{1}{x} - 2x^2 \Big) H_1(x) + 16 \Big(1 - x - 2x^2 \Big) H_0(x) \\
& - 16x \Big(\zeta_2 - 2H_{0,0}(x) - H_2(x) \Big) \Big\} ,
\end{aligned}$$

$$c_{L,g}^{(2)}(x) = \quad (214)$$

$$n_f C_F \Big\{ -\frac{128}{15} + \frac{32}{15x} - \frac{304}{5}x + \frac{336}{5}x^2 - \frac{8}{15} \Big(13 + \frac{4}{x} + 78x - 36x^2 \Big) H_0(x)
\Big\}$$

$$\begin{aligned}
& -8\left(1+3x-4x^2\right)H_1(x)+\frac{32}{15}\left(\frac{1}{x^2}-5x+6x^3\right)H_{-1,0}(x)-\frac{64}{3}x\left(1+\frac{3}{5}x^2\right)H_{0,0}(x) \\
& +\frac{16}{3}x\left(1+\frac{12}{5}x^2\right)\zeta_2-16xH_2(x)\Big\} \\
& +n_f C_A\left\{\frac{16}{3}-\frac{16}{9x}+\frac{272}{3}x-\frac{848}{9}x^2+\frac{16}{3}\left(3-\frac{1}{x}+27x-29x^2\right)H_1(x)-32x(2-x)\zeta_2\right. \\
& +16\left(1+8x-13x^2\right)H_0(x)+32x(1-x)\left(H_{1,0}(x)+H_{1,1}(x)\right)+32x(3-x)H_2(x) \\
& \left.+32x(1+x)H_{-1,0}(x)+96xH_{0,0}(x)\right\}.
\end{aligned}$$

References

- [1] D. J. Gross and F. Wilczek, Phys. Rev. **D8**, 3633 (1973); *ibid.* **D9**, 980 (1974).
H. D. Politzer, Phys. Rev. Lett. **30**, 1346 (1973).
- [2] W. A. Bardeen, A. J. Buras, D. W. Duke, and T. Muta, Phys. Rev. **D18**, 3998 (1978).
- [3] E. G. Floratos, D. A. Ross, and C. T. Sachrajda, Nucl. Phys. **B129**, 66 (1977); *ibid.* **B139**, 545 (1978); *ibid.* **B152**, 493 (1979).
- [4] A. Gonzalez-Arroyo, C. Lopez, and F. J. Ynduráin, Nucl. Phys. **B153**, 161 (1979).
A. Gonzalez-Arroyo and C. Lopez, Nucl. Phys. **B166**, 429 (1980).
C. Lopez and F. J. Ynduráin, Nucl. Phys. **B183**, 157 (1981).
- [5] G. Curci, W. Furmanski, and R. Petronzio, Nucl. Phys. **B175**, 27 (1980).
- [6] W. Furmanski and R. Petronzio, Phys. Lett. **97B**, 437 (1980).
- [7] R. Hamberg and W. L. van Neerven, Nucl. Phys. **B379**, 143 (1992).
R. Hamberg, (1991), Ph.D. thesis, Leiden University.
- [8] R. K. Ellis and W. Vogelsang, (1996), [hep-ph/9602356](#).
- [9] D. W. Duke, J. D. Kimel, and G. A. Sowell, Phys. Rev. **D25**, 71 (1982).
S. N. Coulson and R. E. Ecclestone, Phys. Lett. **115B**, 415 (1982); Nucl. Phys. **B211**, 317 (1983).
A. Devoto, D. W. Duke, J. D. Kimel, and G. A. Sowell, Phys. Rev. **D30**, 541 (1984).
J. L. Miramontes, J. S. Guillen, and E. Zas, Phys. Rev. **D35**, 863 (1987).
- [10] D. I. Kazakov and A. V. Kotikov, Nucl. Phys. **B307**, 721 (1988); *ibid.* **B345**, 299 (1990).
- [11] D. I. Kazakov, A. V. Kotikov, G. Parente, O. A. Sampayo, and J. S. Guillen, Phys. Rev. Lett. **65**, 1535 (1990). *ibid.* **65**, 2921 (1990).
- [12] J. S. Guillen, J. Miramontes, M. Miramontes, G. Parente, and O. A. Sampayo, Nucl. Phys. **B353**, 337 (1991).
- [13] W. L. van Neerven and E. B. Zijlstra, Phys. Lett. **B272**, 127 (1991).
- [14] E. B. Zijlstra and W. L. van Neerven, Phys. Lett. **B273**, 476 (1991); Nucl. Phys. **B383**, 525 (1992).
E. B. Zijlstra, (1993), Ph.D. thesis, Leiden University.
- [15] E. B. Zijlstra and W. L. van Neerven, Phys. Lett. **B297**, 377 (1992).

- [16] S. A. Larin and J. A. M. Vermaseren, Z. Phys. **C57**, 93 (1993).
- [17] S. A. Larin, F. V. Tkachev, and J. A. M. Vermaseren, Phys. Rev. Lett. **66**, 862 (1991).
- [18] S. A. Larin and J. A. M. Vermaseren, Phys. Lett. **B259**, 345 (1991).
- [19] S. A. Larin, P. Nogueira, T. van Ritbergen, and J. A. M. Vermaseren, Nucl. Phys. **B492**, 338 (1997), [hep-ph/9605317](#).
- [20] A. L. Kataev, A. V. Kotikov, G. Parente, and A. V. Sidorov, Phys. Lett. **B417**, 374 (1998), [hep-ph/9706534](#).
J. Santiago and F. J. Yndurain, (1999), [hep-ph/9904344](#); [hep-ph/9907387](#).
W. L. van Neerven and A. Vogt, (1999), [hep-ph/9907472](#).
- [21] Y. Matiounine, J. Smith, and W. L. van Neerven, Phys. Rev. **D57**, 6701 (1998), [hep-ph/9801224](#).
- [22] J. A. M. Vermaseren, Int. J. Mod. Phys. **A14**, 2037 (1999), [hep-ph/9806280](#).
- [23] J. Blümlein and S. Kurth, Phys. Rev. **D60**, 014018 (1999), [hep-ph/9810241](#).
- [24] E. Remiddi and J. A. M. Vermaseren, (1999), [hep-ph/9905237](#).
- [25] A. J. Buras, Rev. Mod. Phys. **52**, 199 (1980).
- [26] E. Reya, Phys. Rept. **69**, 195 (1981).
G. Altarelli, Phys. Rept. **81**, 1 (1982).
CTEQ, R. Brock *et al.*, Rev. Mod. Phys. **67**, 157 (1995).
- [27] G. 't Hooft and M. Veltman, Nucl. Phys. **B44**, 189 (1972).
- [28] C. G. Bollini and J. J. Giambiagi, Nuovo Cim. **12B**, 20 (1972).
J. F. Ashmore, Lett. Nuovo Cim. **4**, 289 (1972).
G. M. Cicuta and E. Montaldi, Nuovo Cim. Lett. **4**, 329 (1972).
- [29] W. E. Caswell, Phys. Rev. Lett. **33**, 244 (1974).
D. R. T. Jones, Nucl. Phys. **B75**, 531 (1974).
- [30] S. A. Larin, T. van Ritbergen, and J. A. M. Vermaseren, Nucl. Phys. **B427**, 41 (1994).
- [31] J. C. Collins, *Renormalization* (Cambridge University Press, Cambridge, 1987), ISBN 0-521-31177-2.
- [32] S. G. Gorishnii, S. A. Larin, and F. V. Tkachev, Phys. Lett. **124B**, 217 (1983).
S. G. Gorishnii and S. A. Larin, Nucl. Phys. **B283**, 452 (1987).
- [33] H. Kluberg-Stern and J. B. Zuber, Phys. Rev. **D12**, 467 (1975).
J. C. Collins, A. Duncan, and S. D. Joglekar, Phys. Rev. **D16**, 438 (1977).
- [34] D. A. Akyeampong and R. Delbourgo, Nuovo Cim. **17A**, 578 (1973).
- [35] S. G. Gorishnii and S. A. Larin, Phys. Lett. **B172**, 109 (1986).
S. A. Larin, Phys. Lett. **B303**, 113 (1993), [hep-ph/9302240](#).
- [36] S. G. Gorishnii, S. A. Larin, L. R. Surguladze, and F. V. Tkachev, Comput. Phys. Commun. **55**, 381 (1989).
- [37] S. A. Larin, F. V. Tkachev, and J. A. M. Vermaseren, NIKHEF-H-91-18.

- [38] K. G. Chetyrkin and F. V. Tkachev, Nucl. Phys. **B192**, 159 (1981).
F. V. Tkachev, Phys. Lett. **100B**, 65 (1981).
- [39] G. Passarino and M. Veltman, Nucl. Phys. **B160**, 151 (1979).
- [40] S. A. Larin, (1981), Diploma thesis, M. V. Lomonosov Physics Dept., Moscow State University.
- [41] J. A. M. Vermaseren, *Symbolic Manipulation with FORM* (Computer Algebra Nederland, Kruislaan 413, 1098 SJ Amsterdam, 1991), ISBN 90-74116-01-9.
- [42] J. M. Borwein, D. M. Bradley, D. J. Broadhurst, and P. Lisonek, (1998), CECM research report 98-106.
- [43] P. Nogueira, J. Comput. Phys. **105**, 279 (1993).
- [44] D. I. Kazakov and A. V. Kotikov, Phys. Lett. **B291**, 171 (1992).
- [45] W. Furmanski and R. Petronzio, Zeit. Phys. **C11**, 293 (1982).
- [46] V. N. Gribov and L. N. Lipatov, Sov. J. Nucl. Phys. **15**, 438 (1972); *ibid.* **15**, 675 (1972).
L. N. Lipatov, Sov. J. Nucl. Phys. **20**, 94 (1975).
G. Altarelli and G. Parisi, Nucl. Phys. **B126**, 298 (1977).
Y. L. Dokshitzer, Sov. Phys. JETP **46**, 641 (1977).
G. Parisi, Phys. Lett. **90B**, 295 (1980).
- [47] S. Moch and J. A. M. Vermaseren, in preparation.
- [48] L. Lewin, *Polylogarithms and Associated Functions* (North Holland, New York, 1981), ISBN 0-444-00550-1.